

QS 015

Mid-Semester Examination

Semester I

Session 2014/2015

1. If $\begin{vmatrix} -2 & 1 & x+2 \\ 3 & x-4 & 5 \\ 0 & 1 & 3 \end{vmatrix} = 25$, find the value of x .
2. Given a complex number $z = \frac{2i}{\sqrt{3}+i}$.
 - (a) State z in the form of $a + bi$, where a and b are real numbers.
 - (b) Find the modulus and argument of z .
3.
 - (a) Simplify $(3\sqrt{2} + 1)^2$ in the form $a + b\sqrt{c}$.
 - (b) If $2 \log x + 3 \log y = 0$, find y in terms of x .
4. Solve $\frac{3x}{x-4} \geq \frac{2x}{7}$.
5. Expand $(1-x)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3 . State the range of x for which the expansion is valid.

Hence,

 - (a) Show that $(9-2x)^{\frac{1}{2}} \approx 3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3 + \dots$
 - (b) By substituting $x = \frac{1}{2}$ in the expansion in part (a), find the value of $\sqrt{8}$ correct to four decimal places.
6. Given a matrix $A = \begin{bmatrix} 1 & 1 & x \\ 3 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$.
 - (a) If $M_{31} = -5$, find the value of x .
 - (b) Hence, find
 - (i) $|A|$.
 - (ii) the values of a , b and c if the cofactor matrix of A is $\begin{bmatrix} -3 & a & -1 \\ b & -9 & 1 \\ -5 & 10 & c \end{bmatrix}$.
 - (iii) A^{-1} .

1. If $\begin{vmatrix} -2 & 1 & x+2 \\ 3 & x-4 & 5 \\ 0 & 1 & 3 \end{vmatrix} = 25$, find the value of x .

Solution

$$\begin{aligned} \begin{vmatrix} -2 & 1 & x+2 \\ 3 & x-4 & 5 \\ 0 & 1 & 3 \end{vmatrix} &= + (0) \begin{vmatrix} 1 & x+2 \\ x-4 & 5 \end{vmatrix} - (1) \begin{vmatrix} -2 & x+2 \\ 3 & 5 \end{vmatrix} + (3) \begin{vmatrix} -2 & 1 \\ 3 & x-4 \end{vmatrix} \\ &= 0 - [(-2)(5) - (3)(x+2)] + 3 [(-2)(x-4) - (3)(1)] \\ &= 0 - [(-10) - (3x+6)] + 3 [(-2x+8) - (3)] \\ &= 0 - [-10 - 3x - 6] + 3 [-2x + 8 - 3] \\ &= 0 - [-16 - 3x] + 3 [-2x + 5] \\ &= 16 + 3x - 6x + 15 \\ &= 31 - 3x \end{aligned}$$

Since $\begin{vmatrix} -2 & 1 & x+2 \\ 3 & x-4 & 5 \\ 0 & 1 & 3 \end{vmatrix} = 25$

$$31 - 3x = 25$$

$$3x = 31 - 25$$

$$3x = 6$$

$$x = 2$$

2. Given a complex number $z = \frac{2i}{\sqrt{3}+i}$.

(a) State z in the form of $a + bi$, where a and b are real numbers.

(b) Find the modulus and argument of z .

Solution

$$\begin{aligned} \text{(a)} \quad z &= \frac{2i}{\sqrt{3}+i} \\ &= \frac{2i}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} \\ &= \frac{2\sqrt{3}i - 2i^2}{3+1} \\ &= \frac{2\sqrt{3}i + 2}{4} \\ &= \frac{2[\sqrt{3}i + 1]}{4} \\ &= \frac{[\sqrt{3}i + 1]}{2} \\ &= \frac{\sqrt{3}i}{2} + \frac{1}{2} \\ &= \frac{1}{2} + \frac{\sqrt{3}i}{2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad |z| &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{3}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{4}{4}} \\ &= 1 \end{aligned}$$

$$\begin{aligned}\alpha &= \tan^{-1} \left[\frac{b}{a} \right] \\ &= \tan^{-1} \left[\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{3}}{2} \cdot \frac{2}{1} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{3}}{1} \cdot \frac{2}{1} \right] \\ &= \tan^{-1} [\sqrt{3}] \\ &= \frac{\pi}{3} \quad \text{or} \quad 1.047 \\ \arg(z), \theta &= \pi - \frac{\pi}{3} \\ &= \frac{2\pi}{3} \quad \text{or} \quad 2.09\end{aligned}$$

3. (a) Simplify $(3\sqrt{2} + 1)^2$ in the form $a + b\sqrt{c}$.
(b) If $2 \log x + 3 \log y = 0$, find y in terms of x .

Solution

$$\begin{aligned} \text{(a)} \quad (3\sqrt{2} + 1)^2 &= (3\sqrt{2} + 1)(3\sqrt{2} + 1) \\ &= (3\sqrt{2})^2 + 2(3\sqrt{2}) + 1 \\ &= 18 + 6\sqrt{2} + 1 \\ &= 19 + 6\sqrt{2} \end{aligned}$$

$$\text{(b)} \quad 2 \log x + 3 \log y = 0$$

$$\log x^2 + \log y^3 = 0$$

$$\log x^2 y^3 = 0$$

$$x^2 y^3 = 1$$

$$y^3 = \frac{1}{x^2}$$

$$y = x^{-\frac{2}{3}}$$

Alternative

$$2 \log x + 3 \log y = 0$$

$$2 \log x = -3 \log y$$

$$\log x^2 = \log y^{-3}$$

$$x^2 = y^{-3}$$

$$y^{-3} = x^2$$

$$(y^{-3})^{-\frac{1}{3}} = (x^2)^{-\frac{1}{3}}$$

$$y = x^{-\frac{2}{3}}$$

4. Solve $\frac{3x}{x-4} \geq \frac{2x}{7}$.

Solution

$$\frac{3x}{x-4} \geq \frac{2x}{7}$$

$$\frac{3x}{x-4} - \frac{2x}{7} \geq 0$$

$$\frac{7(3x) - 2x(x-4)}{7(x-4)} \geq 0$$

$$\frac{21x - 2x^2 + 8x}{7(x-4)} \geq 0$$

$$\frac{29x - 2x^2}{7(x-4)} \geq 0$$

$$\frac{x(29 - 2x)}{7(x-4)} \geq 0$$

Critical Values:

$$x = 0; \quad x = \frac{29}{2}; \quad x = 4$$

	$(-\infty, 0)$	$(0, 4)$	$(4, \frac{29}{2})$	$(\frac{29}{2}, \infty)$
x	-	+	+	+
$(29 - 2x)$	+	+	+	-
$(x - 4)$	-	-	+	+
$\frac{x(29 - 2x)}{7(x - 4)}$	(+)	-	(+)	-

$$\therefore (-\infty, 0] \cup (4, \frac{29}{2}]$$

5. Expand $(1-x)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3 . State the range of x for which the expansion is valid.

Hence,

(a) Show that $(9-2x)^{\frac{1}{2}} \approx 3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3 + \dots$

- (b) By substituting $x = \frac{1}{2}$ in the expansion in part (a), find the value of $\sqrt{8}$ correct to four decimal places.

Solution

$$\begin{aligned}(1-x)^{\frac{1}{2}} &= 1 + \binom{\frac{1}{2}}{1}(-x) + \frac{\binom{\frac{1}{2}}{2}\binom{-1}{2}}{2!}(-x)^2 + \frac{\binom{\frac{1}{2}}{3}\binom{-1}{2}\binom{-3}{2}}{3!}(-x)^3 + \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots\end{aligned}$$

the range of x for which the expansion is valid

$$|x| < 1$$

$$-1 < x < 1$$

$$\begin{aligned}\text{(a) } (9-2x)^{\frac{1}{2}} &= \left[9\left(1-\frac{2}{9}x\right)\right]^{\frac{1}{2}} \\ &= 3\left(1-\frac{2}{9}x\right)^{\frac{1}{2}} \\ &= 3\left[1 - \frac{1}{2}\left(\frac{2}{9}x\right) - \frac{1}{8}\left(\frac{2}{9}x\right)^2 - \frac{1}{16}\left(\frac{2}{9}x\right)^3 + \dots\right] \\ &= 3\left[1 - \frac{1}{9}x - \frac{1}{162}x^2 - \frac{1}{1458}x^3 + \dots\right] \\ &= 3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3 + \dots\end{aligned}$$

$$\text{(b) } (9-2x)^{\frac{1}{2}} = 3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3 + \dots$$

$$\left[9 - 2\left(\frac{1}{2}\right)\right]^{\frac{1}{2}} = 3 - \frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{54}\left(\frac{1}{2}\right)^2 - \frac{1}{486}\left(\frac{1}{2}\right)^3 + \dots$$

$$\sqrt{8} = 2.8284$$

6. Given a matrix $A = \begin{bmatrix} 1 & 1 & x \\ 3 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$.

(a) If $M_{31} = -5$, find the value of x .

(b) Hence, find

(i) $|A|$.

(ii) the values of a , b and c if the cofactor matrix of A is

$$\begin{bmatrix} -3 & a & -1 \\ b & -9 & 1 \\ -5 & 10 & c \end{bmatrix}$$

(iii) A^{-1} .

Solution

$$A = \begin{bmatrix} 1 & 1 & x \\ 3 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

(a) $M_{31} = -5$

$$\begin{vmatrix} 1 & x \\ 2 & 5 \end{vmatrix} = -5$$

$$5 - 2x = -5$$

$$2x = 10$$

$$x = 5$$

(b) (i) $A = \begin{bmatrix} 1 & 1 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$

$$|A| = + (1) \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} - (1) \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} + (5) \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$$

$$|A| = (1) [(2)(1) - (5)(1)] - (1)[(3)(1) - (5)(2)] + (5)[(3)(1) - (2)(2)]$$

$$|A| = (1) [2 - 5] - (1)[3 - 10] + (5)[3 - 4]$$

$$|A| = (1) [-3] - (1)[-7] + (5)[-1]$$

$$|A| = -3 + 7 - 5$$

$$|A| = -1$$

$$\begin{aligned} \text{(b) (ii)} \quad C_{12} &= - \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} \\ &= -[(3)(1) - (5)(2)] \\ &= -[3 - 10] \\ &= 7 \end{aligned}$$

Given that $C_{12} = a$

$$\therefore a = 7$$

$$\begin{aligned} C_{21} &= - \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ &= -[(1)(1) - (5)(1)] \\ &= -[1 - 5] \\ &= 4 \end{aligned}$$

Given that $C_{21} = b$

$$\therefore b = 7$$

$$\begin{aligned} C_{33} &= \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \\ &= -[(1)(2) - (1)(3)] \\ &= -[2 - 3] \\ &= -1 \end{aligned}$$

Given that $C_{33} = c$

$$\therefore c = -1$$

$$(b) (iii) \quad A = \begin{bmatrix} 1 & 1 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} + \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\ - \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ + \begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} & - \begin{vmatrix} 1 & 5 \\ 3 & 5 \end{vmatrix} & + \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 7 & -1 \\ 4 & -9 & 1 \\ -5 & 10 & -1 \end{bmatrix}$$

$$Adj(A) = C^T$$

$$= \begin{bmatrix} -3 & 7 & -1 \\ 4 & -9 & 1 \\ -5 & 10 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 4 & -5 \\ 7 & -9 & 10 \\ -1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -3 & 4 & -5 \\ 7 & -9 & 10 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 5 \\ -7 & 9 & -10 \\ 1 & -1 & 1 \end{bmatrix}$$