

**QS 015**

**Mid-Semester Examination**

**Semester I**

**Session 2013/2014**

1. Evaluate  $\frac{1}{3-\sqrt{5}} - \frac{1}{1+\sqrt{5}}$ .
2. Solve the inequality  $x - 1 < x^2 - 3 \leq 2x + 5$ .
3. If  $\log_a \left( \frac{x}{a^2} \right) = 3 \log_a 2 - \log_a (x - 2a)$ , express x in terms of a.
4. (a) Express  $z = -\sqrt{3} - i$  in polar form.  
(b) Given that the complex number z and its conjugate  $\bar{z}$  satisfy the equation  $z\bar{z} + 2iz = 12 + 6i$ . Find the possible values of z.
5. (a) Given the first term of a geometric series is 40 and its sum to infinity is 60. Find the sum of the first forty terms,  $S_{40}$  of the series.  
(b) Expand  $\frac{1}{(1-2x)^3}$  in ascending powers of x up to the term in  $x^3$  and state the range of x such that the expansion is valid. Hence, approximate  $(0.9)^{-3}$ .
6. Given a matrix  $T = \begin{bmatrix} 1 & 1 & p \\ 0 & -q & 1 \\ 3 & 2 & 1 \end{bmatrix}$ .  
(a) Show that  $|T| = 1 - q(1 - 3p)$ .  
(b) Find the values of p and q if  $|T| = 6$  and the minor for element  $a_{22}$  is -5.  
(c) Hence, determine the inverse of T,  $T^{-1}$  by using the adjoint method.

1. Evaluate  $\frac{1}{3-\sqrt{5}} - \frac{1}{1+\sqrt{5}}$ .

**Solution**

$$\frac{1}{3-\sqrt{5}} - \frac{1}{1+\sqrt{5}}$$
$$= \frac{1}{3-\sqrt{5}} \left( \frac{3+\sqrt{5}}{3+\sqrt{5}} \right) - \frac{1}{1+\sqrt{5}} \left( \frac{1-\sqrt{5}}{1-\sqrt{5}} \right)$$

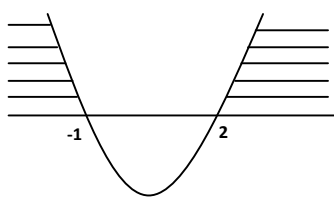
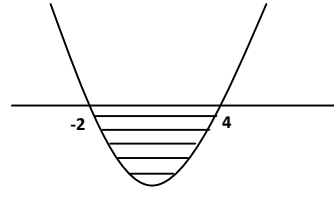
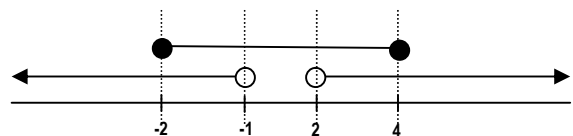
$$= \frac{3+\sqrt{5}}{4} + \frac{1-\sqrt{5}}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

2. Solve the inequality  $x - 1 < x^2 - 3 \leq 2x + 5$ .

**Solution**

$x - 1 < x^2 - 3 \leq 2x + 5$		
$x^2 - 3 > x - 1$ $x^2 - x - 2 > 0$ $(x + 1)(x - 2) > 0$ <div style="text-align: center; margin: 10px 0;">  </div> $x < -1 \text{ or } x > 2$	and	$x^2 - 3 \leq 2x + 5$ $x^2 - 2x - 8 \leq 0$ $(x + 2)(x - 4) \leq 0$ <div style="text-align: center; margin: 10px 0;">  </div> $-2 \leq x \leq 4$
<div style="text-align: center; margin-bottom: 10px;">  </div> $\{x: -2 \leq x < -1\} \cup \{x: 2 < x \leq 4\}$		

3. If  $\log_a \left( \frac{x}{a^2} \right) = 3 \log_a 2 - \log_a(x - 2a)$ , express  $x$  in terms of  $a$ .

**Solution**

$$\log_a \left( \frac{x}{a^2} \right) = 3 \log_a 2 - \log_a(x - 2a)$$

$$\log_a \left( \frac{x}{a^2} \right) = \log_a 2^3 - \log_a(x - 2a)$$

$$\log_a \left( \frac{x}{a^2} \right) = \log_a \left( \frac{8}{x - 2a} \right)$$

$$\left( \frac{x}{a^2} \right) = \left( \frac{8}{x - 2a} \right)$$

$$x^2 - 2ax = 8a^2$$

$$x^2 - 2ax - 8a^2 = 0$$

$$(x - 4a)(x + 2a) = 0$$

$$x = 4a \text{ or } x = -2a$$

Since  $a > 0$ ,  $x \neq -2a$

$$x = 4a$$

4. (a) Express  $z = -\sqrt{3} - i$  in polar form.
- (b) Given that the complex number  $z$  and its conjugate  $\bar{z}$  satisfy the equation  $z\bar{z} + 2iz = 12 + 6i$ . Find the possible values of  $z$ .

**Solution**

$$(a) \quad r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$$

$$\alpha = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\text{argument, } \theta = -\frac{5\pi}{6}$$

*Polar form,*

$$z = 2 \left[ \cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right) \right]$$

$$(b) \quad \text{Let } z = x + yi, \quad \bar{z} = x - yi$$

$$(x + yi)(x - yi) + 2i(x + yi) = 12 + 6i$$

$$x^2 - y^2i^2 + 2xi + 2yi^2 = 12 + 6i$$

$$x^2 + y^2 - 2y + 2xi = 12 + 6i$$

$$x^2 + y^2 - 2y = 12 \quad \dots\dots\dots(1)$$

$$2x = 6 \quad \dots\dots\dots(2)$$

$$x = 3, \quad 9 + y^2 - 2y = 12$$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0$$

$$y = 3, \quad y = -1$$

$$\therefore z = 3 + 3i, \quad z = 3 - i$$

5. (a) Given the first term of a geometric series is 40 and its sum to infinity is 60. Find the sum of the first forty terms,  $S_{40}$  of the series.
- (b) Expand  $\frac{1}{(1-2x)^3}$  in ascending powers of  $x$  up to the term in  $x^3$  and state the range of  $x$  such that the expansion is valid. Hence, approximate  $(0.9)^{-3}$ .

**Solution**

(a)  $S_{\infty} = \frac{a}{1-r}$ ,  $a = 40$

$$\frac{40}{1-r} = 60$$

$$\frac{40}{60} = 1-r$$

$$\therefore r = \frac{1}{3}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{40} = \frac{40 \left[ 1 - \left( \frac{1}{3} \right)^{40} \right]}{1 - \frac{1}{3}}$$

$$S_{40} = 60$$

$$\begin{aligned} \text{(b)} \quad & (1 - 2x)^{-3} \\ &= 1 + (-3)(-2x) + \frac{(-3)(-4)}{2!}(-2x)^2 + \frac{(-3)(-4)(-5)}{3!}(-2x)^3 + \dots \\ &= 1 + 6x + 24x^2 + 80x^3 + \dots \end{aligned}$$

The expansion is valid when

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

$$0.9 = 1 - 2x$$

$$\therefore x = 0.05$$

$$\begin{aligned} [0.9]^{-3} &= [1 - 2(0.05)]^{-3} \\ &= 1 + 6(0.05) + 24(0.05)^2 + 80(0.05)^3 \\ &= 1.37 \end{aligned}$$



6. Given a matrix  $T = \begin{bmatrix} 1 & 1 & p \\ 0 & -q & 1 \\ 3 & 2 & 1 \end{bmatrix}$ .

- (a) Show that  $|T| = 1 - q(1 - 3p)$ .
- (b) Find the values of  $p$  and  $q$  if  $|T| = 6$  and the minor for element  $a_{22}$  is  $-5$ .
- (c) Hence, determine the inverse of  $T$ ,  $T^{-1}$  by using the adjoint method.

**Solution**

- (a) Expansion along first row

$$\begin{aligned} |T| &= a_{11}c_{11} - a_{12}c_{12} + a_{13}c_{13} \\ &= 1 \begin{vmatrix} -q & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + p \begin{vmatrix} 0 & -q \\ 3 & 2 \end{vmatrix} \\ &= (-q - 2) - (0 - 3) + p(0 + 3q) \\ &= -q - 2 + 3 + 3pq \\ &= 1 - q(1 - 3p) \end{aligned}$$

- (b) Given

$$|T| = 6$$

$$1 - q(1 - 3p) = 6$$

$$-q + 3pq = 5 \dots\dots\dots (1)$$

$$\text{Minor, } a_{22} = -5$$

$$\begin{vmatrix} 1 & p \\ 3 & 1 \end{vmatrix} = -5$$

$$1 - 3p = -5$$

$$p = 2$$

$$-q + 3(2)q = 5$$

$$q = 1$$

$$\therefore p = 2, q = 1$$

$$(c) \quad \text{adj } T = C^T$$

$$= \begin{bmatrix} + \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} & + \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} \\ - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \\ + \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & -1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 3 & 3 \\ 3 & -5 & -1 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\text{Inverse of } T, \quad T^{-1} = \frac{1}{|T|} \text{adj } T$$

$$= \frac{1}{6} \begin{bmatrix} -3 & 3 & 3 \\ 3 & -5 & -1 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$