

QS 016

Mid-Semester Examination

Semester I

Session 2010/2011

1. Solve $2\log_3 x = \log_3 2 + \log_3(3x - 4)$.
2. Given $(4 - i)x - 3y = 5 + i$. Find the value of x and y where $x, y \in \mathbb{R}$.
3. If $|x| \geq a$ is defined as $x \geq a$ or $x \leq -a$, then solve $|2x| \geq x^2 - 3$.
4. Given α and β are roots of $x^2 + 3x + 1 = 0$. Find the quadratic equation with roots $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$. **[Out of QS015 syllabus]**
5. The sum of the first 20 terms of an arithmetic sequence is 50, and the sum of the next 20 terms is -50. Find the first term and common difference of the sequence.
6. The function P is defined as $P(x) = px^3 + 2x^2 - qx + 2$ where $p, q \in \mathbb{R}$. $P(x)$ is divisible by $(x + 2)$ and 8 is the remainder when $P(x)$ is divided by $(x + 1)$.
 - (a) Find the value of p and q . Hence, factorise $P(x)$ completely.
 - (b) Express $\frac{2x^2 - 6x + 1}{P(x)}$ in partial fraction. **[Out of QS015 syllabus]**
7.
 - (a) Expand $(4 + x)^{\frac{1}{2}}$ and $(1 + 3x)^{-1}$ in ascending powers of x up to the term in x^2 .
 - (b) By using the result from (a), find the expansion of $\frac{(4+x)^{\frac{1}{2}}}{(1+3x)}$ in ascending powers of x up to the term in x^2 . State the range of x for which the expansion is valid.

1. Solve $2\log_3 x = \log_3 2 + \log_3(3x - 4)$.

Solution

$$2\log_3 x = \log_3 2 + \log_3(3x - 4)$$

$$\log_3 x^2 = \log_3 2(3x - 4)$$

$$x^2 = 2(3x - 4)$$

$$x^2 = 6x - 8$$

$$x^2 - 6x + 8 = 0$$

$$(x - 4)(x - 2) = 0$$

$$x = 4, 2$$

2. Given $(4 - i)x - 3y = 5 + i$. Find the value of x and y where $x, y \in \mathbb{R}$.

Solution

$$(4 - i)x - 3y = 5 + i$$

$$4x - xi - 3y = 5 + i$$

$$(4x - 3y) - xi = 5 + i$$

$$-xi = i \dots\dots\dots (1)$$

$$x = -1$$

$$4x - 3y = 5 \dots\dots\dots (2)$$

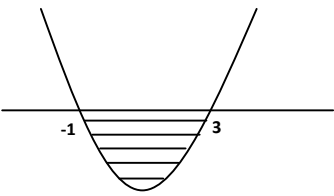
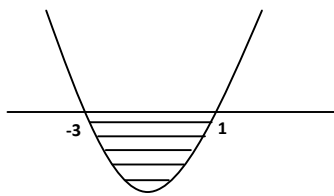
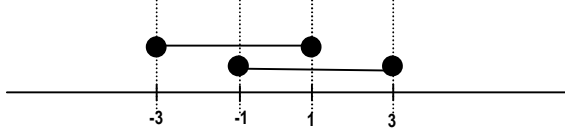
$$4(-1) - 3y = 5$$

$$y = -3$$

$$\therefore x = -1, y = -3$$

3. If $|x| \geq a$ is defined as $x \geq a$ or $x \leq -a$, then solve $|2x| \geq x^2 - 3$.

Solution

$ 2x \geq x^2 - 3$		
$2x \geq x^2 - 3$ $x^2 - 2x - 3 \leq 0$ $(x - 3)(x + 1) \leq 0$ <div style="text-align: center; margin: 10px 0;">  </div> <p style="text-align: center;">$[-1, 3]$</p>	or	$2x \leq -(x^2 - 3)$ $2x \leq -x^2 + 3$ $x^2 + 2x - 3 \leq 0$ $(x - 1)(x + 3) \leq 0$ <div style="text-align: center; margin: 10px 0;">  </div> <p style="text-align: center;">$[-3, 1]$</p>
<div style="text-align: center; margin-bottom: 10px;">  </div> <p style="text-align: center;">$\therefore [-3, 3]$</p>		

4. Given α and β are roots of $x^2 + 3x + 1 = 0$. Find the quadratic equation with roots $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$. **[Out of QS015 syllabus]**

Solution

$$\alpha + \beta = -3, \quad \alpha\beta = 1$$

Form a new equation with roots:

$$\alpha + \frac{1}{\alpha}, \quad \beta + \frac{1}{\beta}$$

Sum of new roots:

$$\begin{aligned} & \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} \\ &= \alpha + \beta + \frac{\alpha + \beta}{\alpha\beta} \\ &= -3 - 3 \\ &= -6 \end{aligned}$$

Product of new roots:

$$\begin{aligned} & \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) \\ &= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} \\ &= 1 + \frac{\alpha^2 + \beta^2}{\alpha\beta} + 1 \\ &= 2 + \frac{\alpha^2 + \beta^2}{\alpha\beta} \\ &= 2 + \frac{\alpha^2 + \beta^2}{1} \\ &= 2 + (\alpha^2 + \beta^2) \\ &= 2 + (\alpha + \beta)^2 - \alpha\beta \\ &= 2 + (-3)^2 - 2 \\ &= 9 \end{aligned}$$

The new equation: $x^2 + 6x + 9 = 0$

5. The sum of the first 20 terms of an arithmetic sequence is 50, and the sum of the next 20 terms is -50. Find the first term and common difference of the sequence.

Solution

$$S_n = \frac{n}{2} [2a + (n - 1)d]$$

$$S_{20} = \frac{20}{2} [2a + (19)d] = 10 [2a + 19d]$$

$$\frac{20}{2} [2a + (19)d] = 50$$

$$2a + 19d = 5 \dots\dots\dots (1)$$

$$S_{40} = \frac{40}{2} [2a + (39)d] = 20 [2a + 39d]$$

$$S_{40} - S_{20} = -50$$

$$20 [2a + 39d] - 50 = -50$$

$$40a + 780d = 0 \dots\dots\dots (2)$$

Solve (1) and (2)

$$d = -\frac{1}{4}$$

$$a = \frac{39}{8}$$

6. The function P is defined as $P(x) = px^3 + 2x^2 - qx + 2$ where $p, q \in \mathbb{R}$. $P(x)$ is divisible by $(x + 2)$ and 8 is the remainder when $P(x)$ is divided by $(x + 1)$.
- (a) Find the value of p and q . Hence, factorise $P(x)$ completely.
- (b) Express $\frac{2x^2 - 6x + 1}{P(x)}$ in partial fraction. **[Out of QS015 syllabus]**

Solution

(a) $P(-2) = 0$
 $-8p + 2q + 10 = 0$ (1)

$P(-1) = 8$
 $-p + q + 4 = 8$ (2)

Solve (1) and (2)

$P = 3, q = 7$

$\therefore P(x) = 3x^3 + 2x^2 - 7x + 2$

Using long division:

$$\begin{array}{r} 3x^2 - 4x + 1 \\ x + 2 \overline{) 3x^3 + 2x^2 - 7x + 2} \end{array}$$

$P(x) = (x + 2)(3x^2 - 4x + 1)$
 $= (x + 2)(3x - 1)(x - 1)$

(b) $\frac{2x^2 - 6x + 1}{(x+2)(3x-1)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(3x-1)} + \frac{C}{(x-1)}$

$2x^2 - 6x + 1 = A(3x - 1)(x - 1) + B(x + 2)(x - 1) + C(x + 2)(3x - 1)$

When $x = 1$: $C = -\frac{1}{2}$

When $x = -2$: $A = 1$

When $x = \frac{1}{3}$: $B = \frac{1}{2}$

$\therefore A = 1; B = \frac{1}{2}; C = -\frac{1}{2}$

(c) $\frac{2x^2 - 6x + 1}{(x+2)(3x-1)(x-1)} = \frac{1}{(x+2)} + \frac{1}{2(3x-1)} - \frac{1}{2(x-1)}$

7. (a) Expand $(4 + x)^{\frac{1}{2}}$ and $(1 + 3x)^{-1}$ in ascending powers of x up to their term in x^2 .
- (b) By using the result from (a), find the expansion of $\frac{(4+x)^{\frac{1}{2}}}{(1+3x)}$ in ascending powers of x up to the term in x^2 . State the range of x for which the expansion is valid.

Solution

$$\begin{aligned}
 \text{(a)} \quad & (4 + x)^{\frac{1}{2}} \\
 &= 4^{\frac{1}{2}} \left(1 + \frac{x}{4}\right)^{\frac{1}{2}} \\
 &= 2 \left[1 + \left(\frac{1}{2}\right) \left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right)}{2!} \left(\frac{x}{4}\right)^2 + \dots \right] \\
 &= 2 \left[1 + \frac{1}{8}x - \frac{1}{128}x^2 + \dots \right] \\
 &= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \dots
 \end{aligned}$$

$$\begin{aligned}
 & (1 + 3x)^{-1} \\
 &= 1 + (-1)(3x) + \frac{(-1)(-2)}{2!} (3x)^2 + \dots \\
 &= 1 - 3x + 9x^2 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \frac{(4+x)^{\frac{1}{2}}}{(1+3x)} \\
 &= (4 + x)^{\frac{1}{2}} (1 + 3x)^{-1} \\
 &= \left[2 + \frac{1}{4}x - \frac{1}{64}x^2 + \dots \right] [1 - 3x + 9x^2 + \dots] \\
 &= 2 - 6x + 18x^2 + \dots + \frac{1}{4}x - \frac{3}{4}x^2 + \dots - \frac{1}{64}x^2 + \dots \\
 &= 2 - \frac{23}{4}x + \frac{1103}{64}x^2
 \end{aligned}$$

$$\left|\frac{x}{4}\right| < 1 \cap |x| < \frac{1}{3}$$

$$|x| < 4 \cap |x| < \frac{1}{3}$$

$$|x| < 4 \cap |x| < \frac{1}{3}$$

Therefore:

$$|x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$