

**QM 016**

**Mid-Semester Examination**

**Semester I**

**Session 2009/2010**

1. (a) Solve  $9^{4x+1} = 27$ .  
(b) Given  $z_1 = 4 - 3i$  and  $z_2 = 3 + 2i$ . Find  $z_1 z_2$  in the form of  $a+bi$ .
2. If  $\alpha$  and  $\beta$  are roots of the equation  $5x^2 - 2x - 9 = 0$ , find the quadratic equation whose roots are  $\alpha^2$  and  $\beta^2$ . **[Out of QS015 Syllabus]**
3. Solve  $|x + 4| > 2|x - 1|$ .
4. The polynomial  $P(x) = 2x^3 - 3ax^2 + ax + b$  has  $(x - 1)$  as a factor and leaves a remainder of  $-54$  when divided by  $(x + 2)$ . Find the values of  $a$  and  $b$ .  
**[QS015 Chapter 6]**
5. Express  $\frac{3}{(x-1)^2(x+1)}$  as partial fractions. **[QS015 Chapter 6]**
6. Given that  $\log_2 x - \log_x 8 + \log_2 2^k + k \log_x 4 = 0$ . If  $y = \log_2 x$ , show that  $y^2 + ky + 2k - 3 = 0$ . Find
  - (a) the set of values of  $k$  for which  $y$  is real.
  - (b) the value of  $x$  when  $k = 6$ .
7. Show that  $(1 + ax)^{-\frac{1}{2}} = 1 - \frac{a}{2}x + \frac{3a^2}{8}x^2 + \dots$   
Given that  $(1 + x^2)(1 + ax)^{-\frac{1}{2}} = 1 + x + bx^2 + \dots$ . Find the values of  $a$  and  $b$ . State the set of values of  $x$  for which the above expansion is valid.

1. (a) Solve  $9^{4x+1} = 27$ .
- (b) Given  $z_1 = 4 - 3i$  and  $z_2 = 3 + 2i$ . Find  $z_1 z_2$  in the form of  $a+bi$ .

**Solution**

(a)  $9^{4x+1} = 27$

$$3^{2(4x+1)} = 3^3$$

$$2(4x + 1) = 3$$

$$8x + 2 = 3$$

$$8x = 1$$

$$x = \frac{1}{8}$$

(b)  $z_1 z_2$

$$= (4 - 3i)(3 + 2i)$$

$$= 12 - 9i + 8i - 6i^2$$

$$= 12 - 9i + 8i + 6$$

$$= 18 - i$$

2. If  $\alpha$  and  $\beta$  are roots of the equation  $5x^2 - 2x - 9 = 0$ , find the quadratic equation whose roots are  $\alpha^2$  and  $\beta^2$ . **[Out of QS015 Syllabus]**

**Solution**

$$5x^2 - 2x - 9 = 0$$

$$x^2 - \frac{2}{5}x - \frac{9}{5} = 0$$

$$\text{Sum of roots } \alpha + \beta = \frac{2}{5}$$

$$\text{Product of roots } \alpha\beta = -\frac{9}{5}$$

$$\alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{2}{5}\right)^2 - 2\left(-\frac{9}{5}\right)$$

$$= \frac{4}{25} + \frac{18}{5}$$

$$= \frac{94}{25}$$

$$\alpha^2\beta^2$$

$$= (\alpha\beta)^2$$

$$= \left(-\frac{9}{5}\right)^2$$

$$= \frac{81}{25}$$

The new quadratic equation is

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$x^2 - \left(\frac{94}{25}\right)x + \left(\frac{81}{25}\right) = 0$$

$$25x^2 - 94x + 81 = 0$$

3. Solve  $|x + 4| > 2|x - 1|$ .

**Solution**

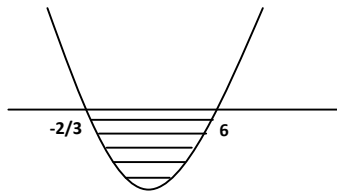
$$|x + 4| > 2|x - 1|$$

$$(x + 4)^2 > [2(x - 1)]^2$$

$$(x^2 + 8x + 16) > (4x^2 - 8x + 4)$$

$$3x^2 - 16x - 12 < 0$$

$$(3x + 2)(x - 6) < 0$$



Solution set:  $\left\{x: \frac{2}{3} < x < 6\right\}$

4. The polynomial  $P(x) = 2x^3 - 3ax^2 + ax + b$  has  $(x - 1)$  as a factor and leaves a remainder of  $-54$  when divided by  $(x + 2)$ . Find the values of  $a$  and  $b$ .  
[QS015 Chapter 6]

**Solution**

$$P(x) = 2x^3 - 3ax^2 + ax + b$$

$$P(1) = 2(1)^3 - 3a(1)^2 + a(1) + b = 0$$

$$2 - 3a + a + b = 0$$

$$2a - b = 2 \quad \dots\dots\dots (1)$$

$$P(-2) = 2(-2)^3 - 3a(-2)^2 + a(-2) + b = -54$$

$$2(-8) - 3a(4) - 2a + b = -54$$

$$14a - b = 38 \quad \dots\dots\dots (2)$$

$$(2) - (1): \quad 12a = 36$$

$$a = 3$$

$$b = 4$$

$$\therefore a = 3, \quad b = 4$$

5. Express  $\frac{3}{(x-1)^2(x+1)}$  as partial fractions. [QS015 Chapter 6]

**Solution**

$$\frac{3}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$3 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

When  $x = 1$ ,

$$3 = 2B$$

$$B = \frac{3}{2}$$

When  $x = -1$

$$3 = 4C$$

$$C = \frac{3}{4}$$

When  $x = 0$

$$3 = -A + \frac{3}{2} + \frac{3}{4}$$

$$A = -\frac{3}{4}$$

$$\therefore A = -\frac{3}{4}, B = \frac{3}{2}, C = \frac{3}{4}$$

$$\frac{3}{(x-1)^2(x+1)} = -\frac{3}{4(x-1)} + \frac{3}{2(x-1)^2} + \frac{3}{4(x+1)}$$

6. Given that  $\log_2 x - \log_x 8 + \log_2 2^k + k \log_x 4 = 0$ . If  $y = \log_2 x$ , show that  $y^2 + ky + 2k - 3 = 0$ . Find
- the set of values of  $k$  for which  $y$  is real.
  - the value of  $x$  when  $k = 6$ .

**Solution**

$$\log_2 x - \log_x 8 + \log_2 2^k + k \log_x 4 = 0$$

$$\log_2 x - \frac{\log_2 8}{\log_2 x} + k \log_2 2 + \frac{k \log_2 4}{\log_2 x} = 0$$

$$\text{Let } y = \log_2 x$$

$$y - \frac{\log_2 2^3}{y} + k(1) + \frac{k \log_2 2^2}{y} = 0$$

$$y - \frac{3 \log_2 2}{y} + k(1) + \frac{2k \log_2 2}{y} = 0$$

$$y - \frac{3(1)}{y} + k(1) + \frac{2k(1)}{y} = 0$$

$$y - \frac{3}{y} + k + \frac{2k}{y} = 0$$

$$y^2 - 3 + ky + 2k = 0$$

$$y^2 + ky + (2k - 3) = 0$$

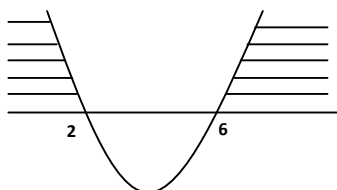
- (a) For real  $y$ :

$$b^2 - 4ac \geq 0$$

$$(k)^2 - 4(1)(2k - 3) \geq 0$$

$$k^2 - 8k + 12 \geq 0$$

$$(k - 2)(k - 6) \geq 0$$



The set of values of  $k$  is  $\{k: k \leq 2 \text{ or } k \geq 6\}$



(c) When  $k = 6$ ,

$$y^2 + 6y + 9 = 0$$

$$(y + 3)^2 = 0$$

$$y + 3 = 0$$

$$y = -3$$

When  $y = -3$

$$\log_2 x = -3$$

$$x = 2^{-3} = \frac{1}{8}$$

7. Show that  $(1 + ax)^{-\frac{1}{2}} = 1 - \frac{a}{2}x + \frac{3a^2}{8}x^2 + \dots$

Given that  $(1 + x^2)(1 + ax)^{-\frac{1}{2}} = 1 + x + bx^2 + \dots$ . Find the values of a and b. State the set of values of x for which the above expansion is valid.

**Solution**

$$\begin{aligned}(1 + ax)^{-\frac{1}{2}} &= 1 + \left(-\frac{1}{2}\right)(ax) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(ax)^2 + \dots \\ &= 1 - \frac{a}{2}x + \frac{3a^2}{8}x^2 + \dots\end{aligned}$$

$$\begin{aligned}(1 + x^2)(1 + ax)^{-\frac{1}{2}} &= (1 + x^2)\left(1 - \frac{a}{2}x + \frac{3a^2}{8}x^2 + \dots\right) \\ &= 1 - \frac{a}{2}x + \frac{3a^2}{8}x^2 + x^2 \dots \\ &= 1 - \frac{a}{2}x + \left(\frac{3a^2}{8} + 1\right)x^2 \dots \\ &= 1 + x + bx^2 + \dots\end{aligned}$$

$$-\frac{a}{2} = 1 \quad \rightarrow \quad a = -2$$

$$b = \frac{3a^2}{8} + 1 \quad \rightarrow \quad b = \frac{5}{2}$$

$$|-2x| < 1$$

$$-\frac{1}{2} < x < \frac{1}{2}$$