KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025/2

Matriculation Programme Examination

Semester II

Session 2014/2015

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1. A survey found that 32% of teenage consumers earned their spending money from working part-time. If five teenagers are selected at random, find the probability that at least two of them are working part-time.

2. Number of accidents at a particular location of a highway occurs at the rate of 1.6 per week. Find the probability
   a. There will be two accidents in a week
   b. There are more than 10 accidents in a five weeks period.

3. Given $P(A \cap B') = 0.25,\ P(A) = 0.48\ \ and\ P(B) = 0.42$. Find $P(A \cap B)$. Is A and B mutually exclusive events? Hence, determine whether A and B are independent events.

4. The following table shows the frequency distribution of the total time (hours) spent by 60 students in a week for revision:

Total time (Hours)	**Number of students**
0 to less than 5	7
5 to less than 10	12
10 to less than 15	15
15 to less than 20	13
20 to less than 25	8
25 to less than 30	5

   Find the mean, mode and standard deviation.

5. The following data are collected from a number of patients X in a clinic and is represented by the stem-and-leaf diagram as below:
   Based on the given diagram,
   a. Find the mode, median, first and third quartiles.
   b. Find the mean and standard deviation given that $\sum x = 1335$ and $\sum x^2 = 71783$.
   c. Calculate Pearson's coefficient of skewness and state the skewness of the data distribution.

6. Seven identical boxes are labeled with numbers 1, 2, 3, 4, 5, 6 and 7. If five boxes are chosen at random,
   a. Find the number of different ways to arrange the boxes in a row such that
      i. There are two odd and three even numbered boxes
      ii. There are only one even numbered box.
   b. Find the probability that there are only two odd numbered boxes next to each other.

7. In a college there are 150 students taking courses in Chemistry, Physics and Biology. Among the students, 92 are females. There are 48 students taking Chemistry which 28 are females. Half of the 68 students taking Physics are females.
   a. Construct the contingency table for the given data.
   b. A student is chosen at random. Find the probability that the student
      i. Takes Biology
      ii. Is a male, given that he takes Biology
      iii. Takes Biology or a female.
   c. Two students are chosen at random, find the probability at least one student is a female and takes Biology.

8. An egg is classified as grade A if it weights at least 100 grams. Suppose eggs lay at a particular farm has the probability of 0.4 being classified as grade A eggs.
   a. If 15 eggs are selected at random from the farm, calculate the probability that more than 20% of them are not grad A eggs.
   b. A retailer bought 500 eggs from the farm.
      i. Approximate the percentage that the retailer would have bought from 220 to 230 grade A eggs.
      ii. If the probability not more than m of the eggs bought are of grade A is 0.9956, determine the value of m.

9. Let *X* be the random variable representing the number obtained when a biased dice is rolled. The probability of the biased dice to give odd numbers is three times higher than even numbers when it is rolled.
   a. If the dice is rolled once,
      i. Construct a probability distribution table for X.

ii. Find the probability of getting a number less than 2.

iii. Find the mean and variance of X.

b. If the dice is rolled 100 times, find the expected value of getting the number "6".

10. The cumulative distribution function of a contimuous random variable, X is given as follows:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{32}x(x+4), & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

a. Calculate P($|X-1| < 1$).

b. Find the median.

c. Determine the probability density function of X. Hence, evaluate $E(3X^2 - 1)$.

END OF QUESTIONS PAPER

1. A survey found that 32% of teenage consumers earned their spending money from working part-time. If five teenagers are selected at random, find the probability that at least two of them are working part-time.

SOLUTION

Let X be the number of teenage consumers working part-time

$$X \sim B(5, 0.32)$$

$$\begin{aligned} P(X \geq 2) &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - \left[{}^5C_0 (0.32)^0 (0.68)^5 + {}^5C_1 (0.32)^1 (0.68)^4 \right] \\ &= 0.5125 \end{aligned}$$

2. Number of accidents at a particular location of a highway occurs at the rate of 1.6 per week. Find the probability

   a. There will be two accidents in a week

   b. There are more than 10 accidents in a five weeks period.

SOLUTION

a) Let X be the number of accidents in one week

$$X \sim P_0(1.6)$$

$$\begin{aligned} P(X=2) &= P(X \geq 2) - P(X \geq 3) \\ &= 0.4751 - 0.2166 \\ &= 0.2585 \end{aligned}$$

b) Let Y be the number of accidents in five weeks

$$\lambda = 5 \times 1.6 = 8$$

$$Y \sim P_0(8)$$

$$\begin{aligned} P(X > 10) &= P(X \geq 11) \\ &= 0.1841 \end{aligned}$$

3. Given $P(A \cap B') = 0.25$, $P(A) = 0.48$ and $P(B) = 0.42$. Find $P(A \cap B)$. Is A and B mutually exclusive events? Hence, determine whether A and B are independent events.

SOLUTION

$P(A \cap B') = 0.25$, $P(A) = 0.48$ and $P(B) = 0.42$
$P(A \cap B') = P(A) - P(A \cap B)$
$0.25 = 0.48 - P(A \cap B)$
$P(A \cap B) = 0.23$
$P(A \cap B) \neq 0$
A and B are not mutually exclusive events

$$P(A) \times P(B) = 0.48 \times 0.42$$
$$= 0.2016$$

$P(A \cap B) \neq P(A) \times P(B)$
A and B are not independent events

4. The following table shows the frequency distribution of the total time (hours) spent by 60 students in a week for revision:

Total time (Hours)	Number of students
0 to less than 5	7
5 to less than 10	12
10 to less than 15	15
15 to less than 20	13
20 to less than 25	8
25 to less than 30	5

Find the mean, mode and standard deviation.

SOLUTION

Total Time	f	x	fx	fx^2
$0 \le x < 5$	7	2.5	17.50	43.75
$5 \le x < 10$	12	7.5	90.00	675.00
$10 \le x < 15$	15	12.5	187.50	2343.75
$15 \le x < 20$	13	17.5	227.50	3981.25
$20 \le x < 25$	8	22.5	180.00	4050.00
$25 \le x < 30$	5	27.5	137.50	3781.25
Total	**60**	**90**	**840**	**14875**

$n = 60$, $\sum fx = 840$, $\sum fx^2 = 14875$

$$\therefore \text{Mean, } \bar{x} = \frac{\sum fx}{n} = \frac{840}{60} = 14$$

$$d_1 = 15 - 12 = 3 \ , \ d_2 = 15 - 13 = 2 \ , \ L_k = 10 \ , \ C = 5$$

$$\therefore \text{Mode } = L_k + \left(\frac{d_1}{d_1 + d_2}\right)C = 10 + \left(\frac{3}{3+2}\right)(5) = 13$$

$$\therefore \text{ Standard deviation, } s = \sqrt{\frac{\sum fx^2 - \frac{\left(\sum fx\right)^2}{n}}{n-1}} = \sqrt{\frac{14875 - \frac{(840)^2}{60}}{60-1}} = 7.266$$

5. The following data are collected from a number of patients X in a clinic and is represented by the stem-and-leaf diagram as below:

   Based on the given diagram,

   a. Find the mode, median, first and third quartiles.
   b. Find the mean and standard deviation given that $\sum x = 1335$ and $\sum x^2 = 71783$.
   c. Calculate Pearson's coefficient of skewness and state the skewness of the data distribution.

SOLUTION

a)

$$\therefore \text{Mode} = 56$$
$$\therefore \text{Median} = 52$$
$$\therefore Q_1 = 36$$
$$\therefore Q_3 = 60$$

b)

$$\sum x = 1335 \quad , \quad \sum x^2 = 71783 \quad , \quad n = 27$$

$$\therefore \text{Mean, } \bar{x} = \frac{\sum x}{n} = \frac{1335}{27} = 49.44$$

$$\therefore \text{ Standard deviation, } \quad s = \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1}} = \sqrt{\frac{71783 - \frac{(1335)^2}{27}}{27-1}} = 14.90$$

c) Pearson coefficient of skewness,

$$\therefore s_k = \frac{3(mean - median)}{s\tan dard\ deviation} = \frac{3(49.44 - 52)}{14.90} = -0.515$$

$\therefore$ Data is skewed to the left

6. Seven identical boxes are labeled with numbers 1, 2, 3, 4, 5, 6 and 7. If five boxes are chosen at random,
   a. Find the number of different ways to arrange the boxes in a row such that
      i. There are two odd and three even numbered boxes
      ii. There are only one even numbered box.
   b. Find the probability that there are only two odd numbered boxes next to each other.

SOLUTION

Odd : 1, 3, 5, 7
Even : 2, 4, 6

a) i) The different ways $= {}^4C_2 \times {}^3C_3 \times 5! = 720$

ii) The different ways $= {}^3C_1 \times {}^4C_4 \times 5! = 360$

b) Let A be only two odd numbered boxes next to each other

$$P(A) = \frac{{}^4C_2 \times {}^3C_3 \times 4! \times 2!}{{}^7P_5} = \frac{4}{35}$$

7. In a college there are 150 students taking courses in Chemistry, Physics and Biology. Among the students, 92 are females. There are 48 students taking Chemistry which 28 are females. Half of the 68 students taking Physics are females.

   a. Construct the contingency table for the given data.
   b. A student is chosen at random. Find the probability that the student
      i. Takes Biology
      ii. Is a male, given that he takes Biology
      iii. Takes Biology or a female.
   c. Two students are chosen at random, find the probability at least one student is a female and takes Biology.

SOLUTION

a)

	C	P	B	T
M	20	34	4	58
F	28	34	30	92
T	48	68	34	150

b

i) $P(B)=\frac{34}{150}=\frac{17}{75}$

ii) $P(M|B)=\frac{4}{34}=\frac{2}{17}$

iii) $P(B\cup F)=P(B)+P(F)-P(B\cap F)$

$$=\frac{34}{150}+\frac{92}{150}-\frac{30}{150}$$

$$=\frac{16}{25}$$

c) Let A be at least one student is a female and takes Biology

$$P(A)=\frac{{}^{30}C_1\times{}^{120}C_1}{{}^{150}C_2}+\frac{{}^{30}C_2\times{}^{120}C_0}{{}^{150}C_2}=\frac{269}{745}$$

8. An egg is classified as grade A if it weights at least 100 grams. Suppose eggs lay at a particular farm has the probability of 0.4 being classified as grade A eggs.
   a. If 15 eggs are selected at random from the farm, calculate the probability that more than 20% of them are not grade A eggs.
   b. A retailer bought 500 eggs from the farm.
      i. Approximate the percentage that the retailer would have bought from 220 to 230 grade A eggs.
      ii. If the probability not more than m of the eggs bought are of grade A is 0.9956, determine the value of m.

SOLUTION

a) Let X be the number of grade A eggs

$X \sim B(15\,,\,0.4)$

20% of 15 eggs = $0.20 \times 15 = 3$

80% of 15 eggs = 0.80 x 15 = 12

More than 20% are not grade A eggs = Less than 80% are grade A eggs

$$\begin{aligned} P(X < 12) &= 1 - P(X \geq 12) \\ &= 1 - 0.0019 \\ &= 0.9981 \end{aligned}$$

b) Let Y be the number of grade A eggs in 500

$$\begin{aligned} &Y \sim B(500\,,\,0.4) \\ &\mu = np = (500)(0.4) = 200 \\ &\sigma^2 = npq = (500)(0.4)(0.6) = 120 \\ &Y \sim N(200\,,\,120) \end{aligned}$$

i) $$\begin{aligned} P(220 \leq Y \leq 230) &= P(219.5 < Y < 230.5) \\ &= P\left(\frac{219.5 - 200}{\sqrt{120}} < Z < \frac{230.5 - 200}{\sqrt{120}}\right) \\ &= P(1.78 < Z < 2.78) \\ &= P(Z > 1.78) - P(Z > 2.78) \\ &= 0.0375 - 0.00272 \\ &= 0.03478 \end{aligned}$$

Percentage $= 0.03478 \times 100\% = 3.478\%$

ii) $P(Y \leq m) = 0.9956$

$P(Y < m + 0.5) = 0.9956$

$$P\left(Z < \frac{m + 0.5 - 200}{\sqrt{120}}\right) = 0.9956$$

$$P\left(Z < \frac{m - 199.5}{\sqrt{120}}\right) = 0.9956$$

$$P\left(Z > \frac{m - 199.5}{\sqrt{120}}\right) = 0.0044$$

$$\frac{m - 199.5}{\sqrt{120}} = 2.62$$

$m = 228.2$

$m = 228$

9. Let *X* be the random variable representing the number obtained when a biased dice is rolled. The probability of the biased dice to give odd numbers is three times higher than even numbers when it is rolled.

    a. If the dice is rolled once,

        i. Construct a probability distribution table for X.

        ii. Find the probability of getting a number less than 2.

        iii. Find the mean and variance of X.

    b. If the dice is rolled 100 times, find the expected value of getting the number "6".

SOLUTION

a i)

X	1	2	3	4	5	6
$P(X=x)$	$3a$	a	$3a$	a	$3a$	a

$$\sum P(X=x)=1$$

$$P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)=1$$
$$3a+a+3a+a+3a+a=1$$
$$12a=1$$
$$a=\frac{1}{12}$$

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$

ii) $P(X<3)=P(X=1)+P(X=2)$
$$=\frac{1}{4}+\frac{1}{12}$$
$$=\frac{1}{3}$$

iii) $$E(X)=\sum x\,P(X=x)$$

$$E(X)=1\left(\frac{1}{4}\right)+2\left(\frac{1}{12}\right)+3\left(\frac{1}{4}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{1}{4}\right)+6\left(\frac{1}{12}\right)$$
$$=\frac{13}{4}$$

$$\boxed{E(X^2) = \sum x^2 \, P(X = x)}$$

$$E(X^2) = 1^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{12}\right) + 3^2\left(\frac{1}{4}\right) + 4^2\left(\frac{1}{12}\right) + 5^2\left(\frac{1}{4}\right) + 6^2\left(\frac{1}{12}\right)$$
$$= \frac{161}{12}$$

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

$$Var(X) = \frac{161}{12} - \left(\frac{13}{4}\right)^2$$
$$= \frac{137}{48}$$

b) $E(X = 6) = 100 \times \frac{1}{12} = \frac{25}{3}$

10. The cumulative distribution function of a contimuous random variable, X is given as follows:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{32}x(x+4), & 0 \le x \le 4 \\ 1, & x \ge 4 \end{cases}$$

a. Calculate P($|X - 1| < 1$).

b. Find the median.

c. Determine the probability density function of X. Hence, evaluate $E(3X^2 - 1)$.

SOLUTION

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{1}{32}x(x+4) & , \quad 0 \le x < 4 \\ 1 & , \quad x \ge 4 \end{cases}$$

a) $P(|X-1| < 1) = P(-1 < X - 1 < 1)$

$$\begin{aligned} &= P(0 < X < 2) \\ &= F(2) - F(0) \\ &= \frac{1}{32}(2)(2+4) - \frac{1}{32}(0)(0+4) \\ &= \frac{3}{8} \end{aligned}$$

b) $F(m) = 0.5$

$\frac{1}{32}m(m+4) = 0.5$

$m^2 + 4m = 16$

$m^2 + 4m - 16 = 0$

$m = 2.47 \quad \text{or} \quad m = -6.47$

Since $0 \le m < 4$

$m = 2.47$

c) $$f(x)=\frac{d}{dx}[F(x)]$$

$x<0,\qquad f(x)=\frac{d}{dx}[0]=0$

$$0\le x<4,\qquad f(x)=\frac{d}{dx}\left[\frac{1}{32}(x^2+4x)\right]$$
$$=\frac{1}{32}(2x+4)$$
$$=\frac{1}{16}(x+2)$$

$x\ge 4,\qquad f(x)=\frac{d}{dx}[1]=0$

$$f(x)=\begin{cases}\frac{1}{16}(x+2), & 0\le x<4\\ 0, & Otherwise\end{cases}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$E(X^2)=\int_0^4 x^2\left[\frac{1}{16}(x+2)\right]dx$$
$$=\frac{1}{16}\int_0^4 x^3+2x^2\,dx$$
$$=\frac{1}{16}\left[\frac{x^4}{4}+\frac{2x^3}{3}\right]_0^4$$
$$=\frac{1}{16}\left[\left(\frac{(4)^4}{4}+\frac{2}{3}(4)^3\right)-(0+0)\right]$$
$$=\frac{20}{3}$$

$$E(aX+b)=aE(X)+b$$

$$E(3X^2-1)=3E(X^2)-1$$
$$=3\left(\frac{20}{3}\right)-1$$
$$=19$$