KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025/1

Matriculation Programme Examination

Semester I

Session 2015/2016

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1. Find the equation of a circle that is passing through points (1, 2), (-1, 2) and (0, -1). Hence, determine its center.

2. Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(1 + e^x)$.

3. Find y in terms of x given that $x\frac{dy}{dx} = (1 - 2x^2)y$ where $x > 0$ and y = 1 when x = 1.

4. Find the general solution of the differential equation $\frac{dy}{dx} + y \cot x = 2 \sin x$.

5. Express $\frac{1-4x}{3+x-2x^2}$ in partial fractions and hence, find the exact value of $\int_0^1 \frac{1-4x}{3+x-2x^2} dx$.

6. (a) Given $f_1(x) = 2x$ and $f_2(x) = -\ln x$.

   i. Without using curve sketching, show that $y = f_1(x)$ and $y = f_2(x)$ intersect on the interval of $[0.1, 1]$.

   ii. Use Newton-Raphson's method to estimate the intersection point of $y = f_1(x)$ and $y = f_2(x)$, with the initial value $x_1 = 1$. Iterate until $|f(x_n)| < 0.005$. Give your answer correct to three decimal places.

   b) By using the tropezoidal rule, find the approximate valur for $\int_0^1 x\sqrt{x+1}\,dx$ when n = 4, correct to four decimal places.

7. (a) If $\boldsymbol{p} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{q} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, show that

$$|\boldsymbol{p} \times \boldsymbol{q}|^2 = |\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}.\boldsymbol{q})^2$$

   (b) Given a triangle ABC with $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{AC} = 3\mathbf{b}$. Use the result in part (a), show that the area of the triangle is $3\sqrt{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$. Hence, deduce the area of the triangle if $\boldsymbol{a} = \boldsymbol{p}$ and $\boldsymbol{b} = \boldsymbol{q}$

8. Given a line $l: x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t,$ and two planes $\pi_1: 2\mathrm{x} - \mathrm{y} + 7\mathrm{z} = 53$ and $\pi_2: 3\mathrm{x} + \mathrm{y} + \mathrm{z} = 1$. Find

   a) The point of intersection between the line l and the plane π_1.

   b) The acute angle between the line l and the plane π_1.

   c) The acute angle between planes π_1 and π_2.

9. (a) Find the equation in standard form of an ellipse which passes through the point (-1, 6) and having foci at (-5, 2) and (3, 2).

   (b) From the result obtained in part (a), sketch the graph of the ellipse.

10. (a) Sketch and shade the region R bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and y – axis. Hence, find the area of the region R.

    (b) If R_1 is a region bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and x-axis, deduce the ratio of $R: R_1$.

    (c) Find the volume of the solid generated when the region R is rotated through 360^0 about the x-axis.

END OF QUESTION PAPER

1. Find the equation of a circle that is passing through points (1, 2), (-1, 2) and (0, -1). Hence, determine its center.

SOLUTION

General Equation of circle:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At (1, 2)

$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$

$2g + 4f + c = -5$ .... (1)

At (-1, 2)

$(-1)^2 + 2^2 + 2g(-1) + 2f(2) + c = 0$

$-2g + 4f + c = -5$ .... (2)

At (0, -1)

$0^2 + (-1)^2 + 2g(0) + 2f(-1) + c = 0$

$0g - 2f + c = -1$ .... (3)

$(1) + (2)$

$8f + 2c = -10$ .... (4)

(3) x 2

$-4f + 2c = -2$ .... (5)

(4) – (5)

$12f = -8$

$f = \frac{-8}{12}$

$f = -\frac{2}{3}$ .... (6)

(6) into (5)

$$-4\left(-\frac{2}{3}\right)+2c=-2$$

$$\frac{8}{3}+2c=-2$$

$$2c=-2-\frac{8}{3}$$

$$2c=-\frac{14}{3}$$

$$c=-\frac{7}{3}$$ .... (7)

Substitute (6) & (7) into (2)

$$-2g+4\left(-\frac{2}{3}\right)-\frac{7}{3}=-5$$

$$-2g-\frac{8}{3}-\frac{7}{3}=-5$$

$$-2g-\frac{15}{3}=-5$$

$$-2g=-5+\frac{15}{3}$$

$$-2g=0$$

$$g=0$$

General Equation of circle:

$$x^2+y^2+2gx+2fy+c=0$$

$$x^2+y^2+2(0)x+2\left(-\frac{2}{3}\right)y-\frac{7}{3}=0$$

$$x^2+y^2-\frac{4}{3}y-\frac{7}{3}=0$$

Center of circle,

$$C = (-g, -f)$$

$$= \left(-0, -\left(-\frac{2}{3}\right)\right)$$

$$= \left(0, \frac{2}{3}\right)$$

2. Show that $\int_1^e x \ln x \, dx = \frac{1}{4}(1 + e^x)$.

SOLUTION

$$\int x \ln x \, dx =$$

$$u = \ln x \qquad\qquad dv = x \, dx$$

$$\frac{du}{dx} = \frac{1}{x} \qquad\qquad \int dv = \int x \, dx$$

$$du = \frac{1}{x} dx \qquad\qquad v = \frac{x^2}{2}$$

$$\int u \, dv = uv - \int v \, du$$

$$\int x \ln x \, dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x} dx\right)$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$\int_1^e x \ln x \, dx = \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_1^e$$

$$= \left[\frac{e^2}{2}(\ln e) - \frac{e^2}{4}\right] - \left[\frac{1^2}{2}(\ln 1) - \frac{1^2}{4}\right]$$

$$= \left[\frac{e^2}{2} - \frac{e^2}{4}\right] - \left[0 - \frac{1}{4}\right]$$

$$= \left[\frac{2e^2}{4} - \frac{e^2}{4}\right] + \frac{1}{4}$$

$$= \frac{e^2 + 1}{4}$$

$$= \frac{1}{4}(1 + e^2)$$

3. Find y in terms of x given that $x\frac{dy}{dx} = (1-2x^2)y$ where $x > 0$ and y = 1 when x = 1.

SOLUTION

$$x\frac{dy}{dx} = (1-2x^2)\mathrm{y}$$

$$\frac{dy}{y} = \frac{(1-2x^2)}{x}\mathrm{dx}$$

$$\frac{dy}{y} = \left(\frac{1}{x} - \frac{2x^2}{x}\right)\mathrm{dx}$$

$$\frac{1}{y}\mathrm{dy} = \left(\frac{1}{x} - 2x\right)\mathrm{dx}$$

$$\int \frac{1}{y}\mathrm{dy} = \int \left(\frac{1}{x} - 2x\right)\mathrm{dx}$$

$$\ln y = \ln x - x^2 + C$$

$$When\ x = 1, y = 1$$

$$\ln 1 = \ln 1 - 1^2 + C$$

$$0 = 0 - 1 + C$$

$$C = 1$$

Particular Solution

$$\ln y = \ln x - x^2 + 1$$

$$\ln y - \ln x = 1 - x^2$$

$$\ln\left(\frac{y}{x}\right) = 1 - x^2$$

$log_a b = c \ \leftrightarrow b = a^c$

$$\frac{y}{x} = e^{1-x^2}$$

$$y = xe^{1-x^2}$$

4. Find the general solution of the differential equation $\frac{dy}{dx} + y\cot x = 2\sin x$.

SOLUTION

$$\frac{dy}{dx} + y\cot x = 2\sin x$$

Compare to

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = cot\ x = \frac{\cos x}{\sin x}$$

$$Q(x) = 2\sin x$$

Integrating factor,

$$V(x) = e^{\int P(x)dx}$$

$$= e^{\int\left(\frac{\cos x}{\sin x}\right)dx}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

$$= e^{\ln(\sin x)}$$

$$= \sin x$$

$$V(x)y = \int V(x)Q(x)dx$$

$$(\sin x)y = \int (\sin x)(2\sin x)dx$$

$$y\sin x = 2\int \sin^2 x\,dx$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$y\sin x = 2\int \frac{1-\cos 2x}{2}dx$$

$$y\sin x = \int 1-\cos 2x\,dx$$

$$y\sin x = x - \frac{\sin 2x}{2} + C$$

5. Express $\frac{1-4x}{3+x-2x^2}$ in partial fractions and hence, find the exact value of $\int_0^1 \frac{1-4x}{3+x-2x^2}\,dx$.

SOLUTION

$$\frac{1-4x}{3+x-2x^2} = \frac{1-4x}{(3-2x)(1+x)}$$

$$\frac{1-4x}{3+x-2x^2} = \frac{A}{(3-2x)} + \frac{B}{(1+x)}$$

$$\frac{1-4x}{3+x-2x^2} = \frac{A(1+x)+B(3-2x)}{(3-2x)(1+x)}$$

$$1-4x = A(1+x)+B(3-2x)$$

$When\ x = -1$

$$1-4(-1) = A(1-1)+B[3-2(-1)]$$

$$5 = 5B$$

$$B = 1$$

$When\ x = \frac{3}{2}$

$$1-4\left(\frac{3}{2}\right) = A\left(1+\frac{3}{2}\right)+B\left[3-2\left(\frac{3}{2}\right)\right]$$

$$-5 = \frac{5}{2}A$$

$$A = -2$$

$$\therefore \frac{1-4x}{3+x-2x^2} = \frac{-2}{(3-2x)} + \frac{1}{(1+x)}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

$$\int \frac{1-4x}{3+x-2x^2}\,dx = \int \frac{-2}{(3-2x)} + \frac{1}{(1+x)}\,dx$$

$$\int \frac{1-4x}{3+x-2x^2}\,dx = \ln(3-2x) + \ln(1+x)$$

$$= \ln(3-2x)(1+x)$$

$$\ln a + \ln b = \ln(ab)$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\int_0^1 \frac{1-4x}{3+x-2x^2}\,dx = [\ln(3-2x)(1+x)]_0^1$$

$$= [\ln(3-2(1))(1+1)] - [\ln(3-2(0))(1+0)]$$

$$= [\ln 2] - [\ln 3]$$

$$= \ln\left(\frac{2}{3}\right)$$

6. (a) Given $f_1(x) = 2\text{x}$ and $f_2(x) = -\ln x$.

   i. Without using curve sketching, show that $y = f_1(x)$ and $y = f_2(x)$ intersect on the interval of $[0.1, 1]$.

   ii. Use Newton-Raphson's method to estimate the intersection point of $y = f_1(x)$ and $y = f_2(x)$, with the initial value $x_1 = 1$. Iterate until $|f(x_n)| < 0.005$. Give your answer correct to three decimal places.

b) By using the tropezoidal rule, find the approximate valur for $\int_0^1 x\sqrt{x+1}\,\text{dx}$ when n = 4, correct to four decimal places.

SOLUTION

(ai)

$f_1(x) = 2\text{x}\,, \qquad f_2(x) = -\ln x$

$Let\ f(x) = f_1(x) - f_2(x)$

$f(x) = 2\text{x} - (-\ln x)$

$f(x) = 2\text{x} + \ln x$

$f(0.1) = 2(0.1) + \ln(0.1) = -2.013 < 0$

$f(1) = 2(1) + \ln 1 = 2 > 0$

Since f(0.1)<0 and f(1)>0, by using intermediate value theorem, there are at least one root in the interval $[0.1, 1]$.

Intermediate Value Theorem

http://mathinsight.org/intermediate_value_theorem_location_roots_refresher

(aii) Newton-Raphson's method

$f(x) = 2\text{x} + \ln x$

$$f'(x) = 2 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2x_n + \ln x_n}{2 + \frac{1}{x_n}}$$

$$x_1 = 1$$

$$x_2 = 1 - \frac{2(1) + \ln(1)}{2 + \frac{1}{(1)}} = 0.3333$$

$$x_3 = 0.3333 - \frac{2(0.3333) + \ln(0.3333)}{2 + \frac{1}{(0.3333)}} = 0.4197$$

$$x_4 = 0.4197 - \frac{2(0.4197) + \ln(0.4197)}{2 + \frac{1}{(0.4197)}} = 0.4263$$

$$x_5 = 0.4263 - \frac{2(0.4263) + \ln(0.4263)}{2 + \frac{1}{(0.4263)}} = 0.4263$$

$\therefore x = 0.426$

$When\ x = 0.426$

$f_1(x) = 2\text{x}$

$f_1(0.426) = 2(0.426) = 0.852$

$\therefore\ (0.426, 0.852)\ is\ the\ intersection\ point\ of\ f_1(x)\ and\ f_2(x)$

(b) **Trapezoidal Rule**

$\int_0^1 x\sqrt{x+1}\,\mathrm{dx}$ when n = 4

$$h = \frac{1-0}{4} = 0.25$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$$h = \frac{b-a}{n}$$

y_n	x_n	$f(x) = x\sqrt{x+1}$	
y_0	0.00	0.00000	
y_1	0.25		0.27951
y_2	0.50		0.61237
y_3	0.75		0.99216
y_4	1.00	1.41421	
	Total	$(y_0 + y_n) = \mathbf{1.41421}$	$(y_1 + y_2 + \cdots + y_{n-1}) = \mathbf{1.88404}$

By trapezoidal rule:

$$\int_0^1 x\sqrt{x+1}\,\mathrm{dx} = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$$= \frac{\mathbf{0.25}}{2}[(\mathbf{1.41421}) + 2(\mathbf{1.88404})]$$

$$= 0.64779$$

$$= 0.6478$$

7. (a) If $\boldsymbol{p} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{q} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, show that

$$|\boldsymbol{p} \times \boldsymbol{q}|^2 = |\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}.\boldsymbol{q})^2$$

(b) Given a triangle ABC with $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{AC} = 3\mathbf{b}$. Use the result in part (a), show that the area of the triangle is $3\sqrt{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$. Hence, deduce the area of the triangle if $\boldsymbol{a} = \boldsymbol{p}$ and $\boldsymbol{b} = \boldsymbol{q}$

SOLUTION

(a) $\boldsymbol{p} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ $\quad$ $\boldsymbol{q} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$$\boldsymbol{p} \times \boldsymbol{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= [(1) - (4)]\boldsymbol{i} - [(-3) - (4)]\boldsymbol{j} + [(6) - (-2)]\boldsymbol{k}$$

$$= -3\boldsymbol{i} - (-7)\boldsymbol{j} + (8)\boldsymbol{k}$$

$$= -3\boldsymbol{i} + 7\boldsymbol{j} + 8\boldsymbol{k}$$

$$|\boldsymbol{p} \times \boldsymbol{q}| = \sqrt{(-3)^2 + (7)^2 + (8)^2}$$

$$= \sqrt{122}$$

$$|\boldsymbol{p} \times \boldsymbol{q}|^2 = 122$$

$$|\boldsymbol{p}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$$

$$= \sqrt{14}$$

$$|\boldsymbol{p}|^2 = 14$$

$$|\boldsymbol{q}| = \sqrt{(2)^2 + (2)^2 + (-1)^2}$$

$$= \sqrt{9}$$

$$|\boldsymbol{q}|^2 = 9$$

$\boldsymbol{p}\,.\,\boldsymbol{q} = (3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}).\,(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$

$= (3)(2) + (-1)(2) + (2)(-1)$

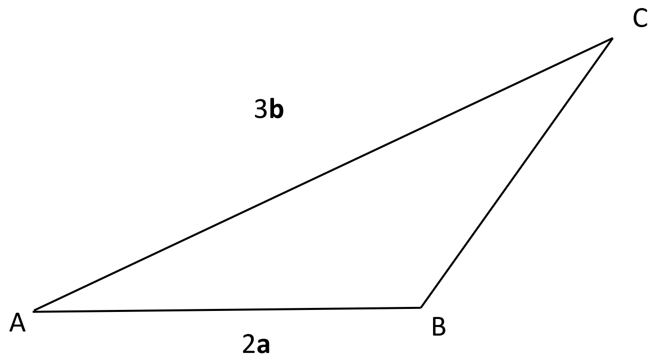
$\boldsymbol{p}\,.\,\boldsymbol{q} = 2$

$|\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}.\boldsymbol{q})^2 = (14)(9) - (2)^2$

$= 122$

$\therefore |\boldsymbol{p} \text{ x } \boldsymbol{q}|^2 = |\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}.\boldsymbol{q})^2$

(b)



$\boldsymbol{Area, A = \frac{1}{2}|\overrightarrow{AB}\ x\ \overrightarrow{AC}|}$

$A = \frac{1}{2}|2\boldsymbol{a}\ x\ 3\boldsymbol{b}|$

$(m\boldsymbol{a}\ x\ \boldsymbol{b}) = m(\boldsymbol{a}\ x\ \boldsymbol{b})$

$2\boldsymbol{a}\ x\ 3\boldsymbol{b} = (2)(\boldsymbol{a}\ x\ \boldsymbol{3}\ \boldsymbol{b})$

$= (2x3)(\boldsymbol{a}\ x\ \ \boldsymbol{b})$

$= (6)(\boldsymbol{a}\ x\ \ \boldsymbol{b})$

$= 3|\boldsymbol{a}\ x\ \boldsymbol{b}|$

$= 3|\boldsymbol{a}\ x\ \boldsymbol{b}|$

$= 3\sqrt{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$

$When\ \boldsymbol{a} = \boldsymbol{p}\ \ and\ \ \boldsymbol{b} = \boldsymbol{q}$

Area = $3\sqrt{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2}$

= $3\sqrt{|\boldsymbol{p}|^2|\boldsymbol{q}|^2 - (\boldsymbol{p}.\boldsymbol{q})^2}$

= $3\sqrt{122}$

8. Given a line $l: x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t,$ and two planes $\pi_1: 2x - y + 7z = 53$ and $\pi_2: 3x + y + z = 1$. Find

   a) The point of intersection between the line l and the plane π_1.

   b) The acute angle between the line l and the plane π_1.

   c) The acute angle between planes π_1 and π_2.

SOLUTION

$l: x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t,$

$\pi_1: 2x - y + 7z = 53$

$\pi_2: 3x + y + z = 1$

(a)

$x = 2 - t, \quad y = -3 + 4t, \quad z = -5 - 3t$ .... (1)

$2x - y + 7z = 53$ .... (2)

Substitute (1) into (2)

$$2(2 - t) - (-3 + 4t) + 7(-5 - 3t) = 53$$

$$4 - 2t + 3 - 4t - 35 - 21t = 53$$

$$-28 - 27t = 53$$

$$27t = -81$$

$$t = -3$$

$When\ t = -3$

$$x = 2 - t = 2 + 3 = 5$$

$$y = -3 + 4t = -3 - 12 = -15$$

$$z = -5 - 3t = -5 + 9 = 4$$

$\therefore$ *the intersection point is* $(5, -15, 4)$

$(\boldsymbol{b})$

$x = 2 - t, \qquad y = -3 + 4t, \quad z = -5 - 3t$ .... (1)

$2x - y + 7z = 53$ .... (2)

Direction vector of the line:

$$\boldsymbol{v} = -\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$$

Normal vector of the plane:

$$\boldsymbol{n} = 2\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$$

Angle Between The Line l And The Plane π_1

$$\cos\theta = \frac{\boldsymbol{v}.\boldsymbol{n}}{|\boldsymbol{v}|.|\boldsymbol{n}|}$$

$$\boldsymbol{v}.\boldsymbol{n} = (-\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}).(2\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k})$$

$$= (-1)(2) + (4)(-1) + (-3)(7)$$

$$= -27$$

$$|\boldsymbol{v}| = \sqrt{(-1)^2 + (4)^2 + (-3)^2}$$

$$= \sqrt{26}$$

$$|\boldsymbol{n}| = \sqrt{(2)^2 + (-1)^2 + (7)^2}$$

$$= \sqrt{54}$$

$$\cos\theta = \frac{\boldsymbol{v}.\boldsymbol{n}}{|\boldsymbol{v}|.|\boldsymbol{n}|}$$

$$= \frac{-27}{\sqrt{26}.\sqrt{54}}$$

$$= -0.7206$$

$$\theta = cos^{-1}\frac{\boldsymbol{v}.\boldsymbol{n}}{|\boldsymbol{v}|.|\boldsymbol{n}|}$$

$$= cos^{-1}(-0.7206)$$

$$= 136.10°$$

Acute Angle Between The Line l And The Plane π_1

$$\alpha = 136.10° - 90°$$

$$= 46.10°$$

$(\boldsymbol{c})$

$\pi_1: 2x - y + 7z = 53$ $\pi_2: 3x + y + z = 1$

Angle Between The Plane π_1 *And The Plane* π_2

$$\theta = cos^{-1}\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}|.|\boldsymbol{n_2}|}$$

$\boldsymbol{n_1} = 2\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$

$\boldsymbol{n_2} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (2\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k})(3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$= (2)(3) + (-1)(1) + (7)(1)$

$= 12$

$|\boldsymbol{n_1}| = \sqrt{(2)^2 + (-1)^2 + (7)^2}$

$= \sqrt{54}$

$|\boldsymbol{n_2}| = \sqrt{(3)^2 + (1)^2 + (1)^2}$

$= \sqrt{11}$

$Angle\ Between\ The\ Plane\ \ \pi_1\ And\ The\ Plane\ \ \pi_2$

$$\theta = cos^{-1}\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}|.|\boldsymbol{n_2}|}$$

$$= cos^{-1}\frac{\mathbf{12}}{\sqrt{54}.\sqrt{11}}$$

$$= cos^{-1}(0.4924)$$

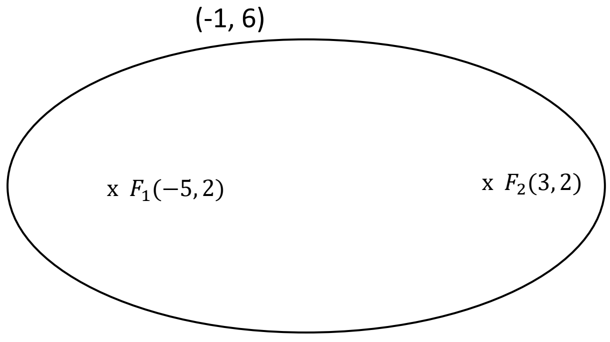
$$= 60.50°$$

9. (a) Find the equation in standard form of an ellipse which passes through the point (-1, 6) and having foci at (-5, 2) and (3, 2).

(b) From the result obtained in part (a), sketch the graph of the ellipse.

SOLUTION

(a)



$Equation\ of\ the\ ellipse:$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$where\ a^2-b^2=c^2$$

$$Centre, C(h,k)=\left(\frac{-5+3}{2},\frac{2+2}{2}\right)$$

$$=\left(\frac{-2}{2},\frac{4}{2}\right)$$

$$=(-1,2)$$

$\boldsymbol{Distance\ between\ two\ foci}$:

$$2c=3-(-5)$$

$$=8$$

$$c=4$$

$a^2 - b^2 = c^2$

$a^2 - b^2 = 4^2$

$a^2 - b^2 = 16$ .... (1)

Equation of the ellipse:

$$\frac{(x+1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

$At\ (-1, 6)$

$$\frac{(-1+1)^2}{a^2} + \frac{(6-2)^2}{b^2} = 1$$

$$0 + \frac{16}{b^2} = 1$$

$b^2 = 16$ .... (2)

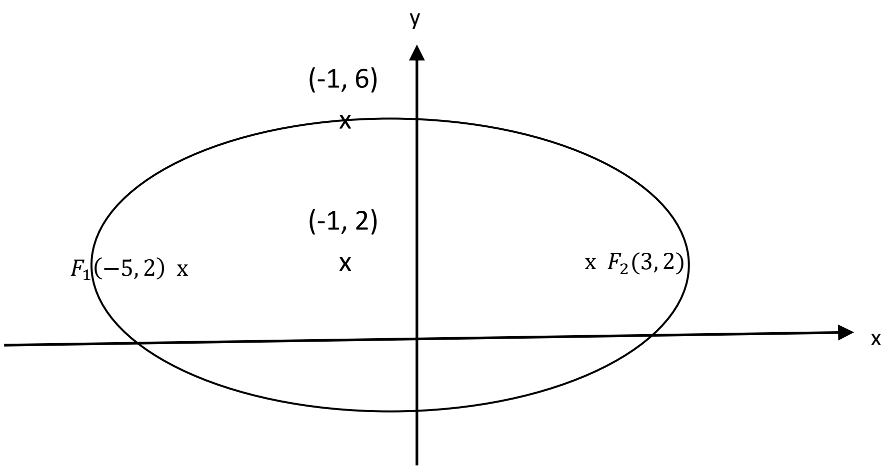
Substitute (2) *into* (1)

$$a^2 - 16 = 16$$

$$a^2 = 32$$

$$\therefore \frac{(x+1)^2}{32} + \frac{(y-2)^2}{16} = 1$$

(b)



10. (a) Sketch and shade the region R bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and y – axis. Hence, find the area of the region R.

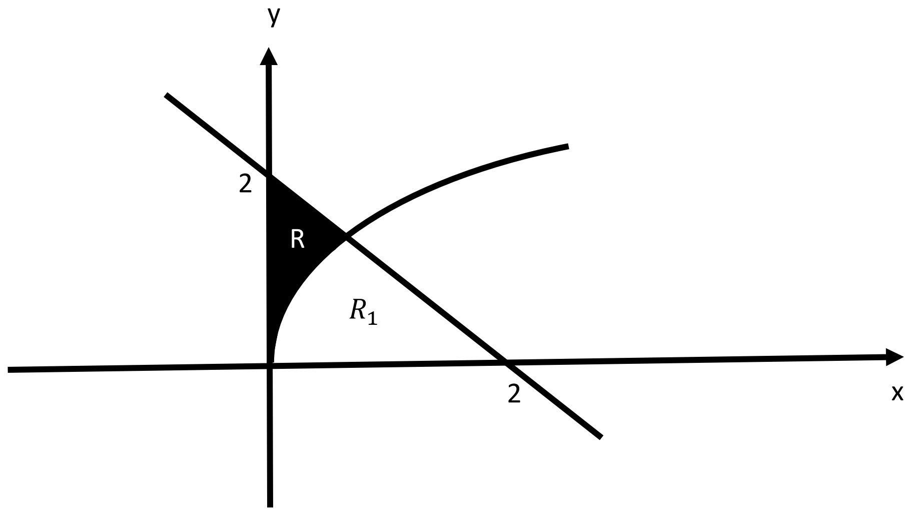
(b) If R_1 is a region bounded by the curve $y = \sqrt{x}$, line $y = 2 - x$ and x-axis, deduce the ratio of $R: R_1$.

(c) Find the volume of the solid generated when the region R is rotated through 360^0 about the x-axis.

SOLUTION

(a)

$y = \sqrt{x}, \qquad y = 2 - x$



$y = \sqrt{x}$ .... (1)

$y = 2 - x$ .... (2)

$Substitute\ (1) into\ (2)$

$\sqrt{x} = 2 - x$

$x = (2 - x)^2$

$x = 4 + x^2 - 4x$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 1 \; or \; x = 4$

$\therefore x = 1$

$$Area, R = \int_0^1 (2-x) - \sqrt{x}\, dx$$

$$= \int_0^1 2 - x - x^{\frac{1}{2}}\, dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_0^1$$

$$= \left[2(1) - \frac{1^2}{2} - \frac{2(1)^{\frac{3}{2}}}{3}\right] - \left[2(0) - \frac{0^2}{2} - \frac{2(0)^{\frac{3}{2}}}{3}\right]$$

$$= \left[2 - \frac{1}{2} - \frac{2}{3}\right] - 0$$

$$= \frac{5}{6}$$

(b)

$$Area\ R_1 = \frac{1}{2}(2)(2) - \frac{5}{6}$$

$$= \frac{7}{6}$$

$\boldsymbol{Ratio\ of\ R{:}\,R_1}$

$$\frac{\boldsymbol{R}}{\boldsymbol{R_1}} = \frac{\frac{5}{6}}{\frac{7}{6}}$$

$$= \frac{5}{6}\, x\, \frac{6}{7}$$

$$= \frac{5}{7}$$

$\therefore \boldsymbol{R{:}\,R_1 = 5{:}\,7}$

(c)

Volume of the solid generated when the region R is rotated through 360^0 about the x-axis.

$$Volume, V = \pi \int_0^1 (2-x)^2 - \left(\sqrt{x}\right)^2 \, dx$$

$$= \pi \int_0^1 (4 + x^2 - 4x) - (x) \, dx$$

$$= \pi \int_0^1 (x^2 - 5x + 4) \, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^1$$

$$= \pi \left[\left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right) - \left(\frac{(0)^3}{3} - \frac{5(0)^2}{2} + 4(0)\right)\right]$$

$$= \pi \left[\left(\frac{1}{3} - \frac{5}{2} + 4\right) - 0\right]$$

$$= \frac{11}{6}\pi$$