



QS 015/2
Matriculation Programme Examination
Semester I
Session 2017/2018

- Express $\frac{3x^2-5}{(x-3)(x^2+2)}$ in partial fractions.
- Solve the equation $\cos \theta + \cos 5\theta = 2 \cos 3\theta$ for $0 \leq \theta \leq \pi$. Give your answers in terms of π .
- Evaluate the following limits:

a. $\lim_{x \rightarrow 2} \frac{x^3-8}{x^2-2x}$

b. $\lim_{x \rightarrow \infty} \sqrt{\frac{5x+7}{6x-5}}$

- Given $y = e^{-2x} \sin 3x$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Hence, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$.

- Given the polynomial $P(x) = x^2 - 4$ and $Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$.
 - Find all zeroes of $P(x)$.
 - When $Q(x)$ is divided by $P(x)$, the remainder is $14x + 52$. Use the remainder theorem to find the values of α and β .
 - Using the values of α and β obtained from part 5(b), find the remainder when $2Q(x) + x$ is divided by $P(x)$.

- Express $\cos \theta + \sqrt{2} \sin \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and α is an acute angle.

Hence,

- Solve the equation $\cos \theta + \sqrt{2} \sin \theta = \frac{\sqrt{3}}{2}$ by giving all solutions between 0° and 360° .

- Show the greatest value of $\frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5}$ is $\frac{5+\sqrt{3}}{22}$.

- State the conditions for continuity of $f(x)$ at $x = a$.
 - By using the conditions for continuity of $f(x)$ at $x = a$, find the values of m and n such that

$$f(x) = \begin{cases} n - 2 \cos x, & x < 0 \\ 2 + mx^2, & 0 \leq x < 2 \\ m - x, & x \geq 2 \end{cases}$$

is continuous on the interval $(-\infty, \infty)$.

- b. If $m = -2$ and $n = 4$, determine whether $f(x)$ is differentiable at $x = 2$ or not.
8. A curve with equation $x^2 - 3y^2 = ae^{y-2x} + by - 6$, where a and b are constants, passes through the point $(1, 2)$.
- a. Given $\frac{dy}{dx} = 1$ at $(1, 2)$, determine the values a and b .
- b. Evaluate $\frac{d^2y}{dx^2}$ at $(1, 2)$.
9. The function f is defined by $f(x) = \frac{\ln(x-1)}{x-1}$ for $x > 1$.
- a. By considering the first and second derivatives of $f(x)$, show that there is only one maximum point on the graph $y = f(x)$.
- b. Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.
- c. Find the x -coordinate of the function f when $\frac{d^2y}{dx^2} = 0$.
10. A curve is defined by the parametric equations $x = 3t - \frac{1}{t}$ and $y = t + \frac{3}{t}$, where $t \neq 0$.
- a. Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$.
- b. Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that
- $$-3 < \frac{dy}{dx} < \frac{1}{3}.$$

END OF QUESTION PAPER

1. Express $\frac{3x^2-5}{(x-3)(x^2+2)}$ in partial fractions.

SOLUTION

$$\begin{aligned}\frac{3x^2-5}{(x-3)(x^2+2)} &= \frac{A}{x-3} + \frac{Bx+C}{x^2+2} \\ &= \frac{A(x^2+2) + (Bx+C)(x-3)}{(x-3)(x^2+2)}\end{aligned}$$

$$3x^2 - 5 = A(x^2 + 2) + (Bx + C)(x - 3)$$

When $x = 3$

$$3(3)^2 - 5 = A[(3)^2 + 2] + [B(3) + C][(3) - 3]$$

$$22 = 11A$$

$$A = 2$$

When $x = 0$

$$3(0)^2 - 5 = (2)[(0)^2 + 2] + [B(0) + C][(0) - 3]$$

$$-5 = 4 - 3C$$

$$3C = 9$$

$$C = 3$$

When $x = 1$

$$3(1)^2 - 5 = (2)[(1)^2 + 2] + [B(1) + (3)][(1) - 3]$$

$$-2 = 6 - 2[B + 3]$$

$$2[B + 3] = 8$$

$$B + 3 = 4$$

$$B = 1$$

$$\frac{3x^2-5}{(x-3)(x^2+2)} = \frac{2}{x-3} + \frac{x+3}{x^2+2}$$

2. Solve the equation $\cos \theta + \cos 5\theta = 2 \cos 3\theta$ for $0 \leq \theta \leq \pi$. Give your answers in terms of π .

SOLUTION

$$0 \leq \theta \leq \pi$$

$$\cos \theta + \cos 5\theta = 2 \cos 3\theta$$

$$\cos 5\theta + \cos \theta = 2 \cos 3\theta$$

$$2 \cos \left(\frac{5\theta + \theta}{2} \right) \cos \left(\frac{5\theta - \theta}{2} \right) = 2 \cos 3\theta$$

$$2 \cos 3\theta \cos 2\theta = 2 \cos 3\theta$$

$$2 \cos 3\theta \cos 2\theta - 2 \cos 3\theta = 0$$

$$\cos 3\theta [2 \cos 2\theta - 2] = 0$$

$$\cos 3\theta = 0$$

$$0 \leq \theta \leq \pi$$

$$0 \leq 3\theta \leq 3\pi$$

$$3\theta = \frac{\pi}{2}, 2\pi - \frac{\pi}{2}, 2\pi + \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

$$2 \cos 2\theta - 2 = 0$$

$$\cos 2\theta = 1$$

$$0 \leq 2\theta \leq 2\pi$$

$$2\theta = 0, \pi$$

$$\theta = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$$

Factor Formulae

$$\text{a) } \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\text{b) } \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

$$\text{c) } \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

$$\text{d) } \cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

3. Evaluate the following limits:

a. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 2x}$

b. $\lim_{x \rightarrow \infty} \sqrt{\frac{5x+7}{6x-5}}$

SOLUTION

a.
$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 2x} &= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 + 2x + 4)}{x} \\ &= \frac{2^2 + 2(2) + 4}{2} \\ &= 6\end{aligned}$$

b.
$$\begin{aligned}\lim_{x \rightarrow \infty} \sqrt{\frac{5x+7}{6x-5}} &= \lim_{x \rightarrow \infty} \sqrt{\frac{\frac{5x}{x} + \frac{7}{x}}{\frac{6x}{x} - \frac{5}{x}}} \\ &= \lim_{x \rightarrow \infty} \sqrt{\frac{5 + \frac{7}{x}}{6 - \frac{5}{x}}} \\ &= \sqrt{\frac{5 + 0}{6 - 0}} \\ &= \sqrt{\frac{5}{6}}\end{aligned}$$

4. Given $y = e^{-2x} \sin 3x$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Hence, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$.

SOLUTION

$$y = e^{-2x} \sin 3x$$

$$u = e^{-2x}$$

$$v = \sin 3x$$

$$u' = -2e^{-2x}$$

$$v' = 3 \cos 3x$$

$$\frac{dy}{dx} = uv' + vu'$$

$$= (e^{-2x})(3 \cos 3x) + (\sin 3x)(-2e^{-2x})$$

$$= 3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x$$

$$\frac{dy}{dx} = e^{-2x}(3 \cos 3x - 2 \sin 3x)$$

$$u = e^{-2x}$$

$$v = 3 \cos 3x - 2 \sin 3x$$

$$u' = -2e^{-2x}$$

$$v' = -9 \sin 3x - 6 \cos 3x$$

$$\frac{d^2y}{dx^2} = uv' + vu'$$

$$= (e^{-2x})(-9 \sin 3x - 6 \cos 3x) + (3 \cos 3x - 2 \sin 3x)(-2e^{-2x})$$

$$= (e^{-2x})[-9 \sin 3x - 6 \cos 3x - 2(3 \cos 3x - 2 \sin 3x)]$$

$$= (e^{-2x})[-9 \sin 3x - 6 \cos 3x - 6 \cos 3x + 4 \sin 3x]$$

$$= (e^{-2x})[-5 \sin 3x - 12 \cos 3x]$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y$$

$$= (e^{-2x})[-5 \sin 3x - 12 \cos 3x] + 4[3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x] + 13[e^{-2x} \sin 3x]$$

$$= -5e^{-2x} \sin 3x - 12 \cos e^{-2x} 3x + 12e^{-2x} \cos 3x - 8e^{-2x} \sin 3x + 13e^{-2x} \sin 3x$$

$$= 0$$

5. Given the polynomial $P(x) = x^2 - 4$ and $Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$.
- Find all zeroes of $P(x)$.
 - When $Q(x)$ is divided by $P(x)$, the remainder is $14x + 52$. Use the remainder theorem to find the values of α and β .
 - Using the values of α and β obtained from part 5(b), find the remainder when $2Q(x) + x$ is divided by $P(x)$.

SOLUTION

$$P(x) = x^2 - 4$$

$$Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$$

(a) When $P(x) = 0$

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm\sqrt{4}$$

$$= \pm 2$$

Therefore, zeros are $-2, 2$

(b) $D(x) = P(x) = x^2 - 4 = (x + 2)(x - 2)$

$$Q(x) = \alpha x^4 + x^3 + 2x^2 + \beta x + 28$$

$$R(x) = 14x + 52$$

Remainder Theorem

When a polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$

$$\alpha x^4 + x^3 + 2x^2 + \beta x + 28 = Q(x)(x^2 - 4) + (14x + 52)$$

$$\alpha x^4 + x^3 + 2x^2 + \beta x + 28 = Q(x)(x + 2)(x - 2) + (14x + 52)$$

When $x = 2$

$$Q(2) = R(2)$$

$$\alpha(2)^4 + (2)^3 + 2(2)^2 + \beta(2) + 28 = [14(2) + 52]$$

$$16\alpha + 16 + 2\beta + 28 = [28 + 52]$$

$$16\alpha + 2\beta = 36$$

$$8\alpha + \beta = 18 \quad \dots\dots\dots (1)$$

When $x = -2$

$$Q(-2) = R(-2)$$

$$\alpha(-2)^4 + (-2)^3 + 2(-2)^2 + \beta(-2) + 28 = [14(-2) + 52]$$

$$16\alpha - 8 + 8 - 2\beta + 28 = [-28 + 52]$$

$$16\alpha - 2\beta + 28 = 24$$

$$16\alpha - 2\beta = -4$$

$$8\alpha - \beta = -2 \quad \dots\dots\dots (2)$$

$$(1) + (2)$$

$$16\alpha = 16$$

$$\alpha = 1$$

$$8 - \beta = -2$$

$$\beta = 10$$

$$\therefore \alpha = 1, \quad \beta = 10$$

(c) $Q(x) = x^4 + x^3 + 2x^2 + 10x + 28$

$$P(x) = 2Q(x) + x$$

$$D(x) = x^2 - 4$$

$$(x^4 + x^3 + 2x^2 + 10x + 28) = [Q(x)(x + 2)(x - 2) + (14x + 52)]$$

$$2(x^4 + x^3 + 2x^2 + 10x + 28) = 2[Q(x)(x + 2)(x - 2) + (14x + 52)]$$

$$2[x^4 + x^3 + 2x^2 + 10x + 28] + x = 2[Q(x)(x + 2)(x - 2) + (14x + 52)] + x$$

$$2[x^4 + x^3 + 2x^2 + 10x + 28] + x = 2Q(x)(x + 2)(x - 2) + 2(14x + 52) + x$$

$$R(x) = 2(14x + 52) + x$$

$$= 28x + 104 + x$$

$$= 29x + 104$$

6. Express $\cos \theta + \sqrt{2} \sin \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and α is an acute angle.

Hence,

- a. Solve the equation $\cos \theta + \sqrt{2} \sin \theta = \frac{\sqrt{3}}{2}$ by giving all solutions between 0° and 360° .
- b. Show the greatest value of $\frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5}$ is $\frac{5 + \sqrt{3}}{22}$.

SOLUTION

$$\cos \theta + \sqrt{2} \sin \theta = R \sin(\theta + \alpha)$$

$$\cos \theta + \sqrt{2} \sin \theta = R(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$$

$$\cos \theta + \sqrt{2} \sin \theta = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$$

$$\cos \theta + \sqrt{2} \sin \theta = R \cos \theta \sin \alpha + R \sin \theta \cos \alpha$$

$$R \cos \theta \sin \alpha = \cos \theta$$

$$R \sin \alpha = 1 \quad \dots\dots\dots (1)$$

$$R \sin \theta \cos \alpha = \sqrt{2} \sin \theta$$

$$R \cos \alpha = \sqrt{2} \quad \dots\dots\dots (2)$$

$$(1)^2 + (2)^2$$

$$(R^2 \sin^2 \alpha) + (R^2 \cos^2 \alpha) = 1^2 + (\sqrt{2})^2$$

$$R^2(\sin^2 \alpha + \cos^2 \alpha) = 3$$

$$R^2 = 3$$

$$R = \sqrt{3}$$

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$(1) \div (2)$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{\sqrt{2}}$$

$$\tan \alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = 35.26^\circ$$

$$\cos \theta + \sqrt{2} \sin \theta = \sqrt{3} \sin(\theta + 35.26^\circ)$$

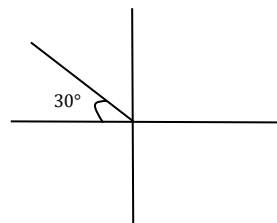
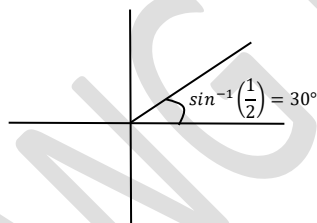
$$(a) \cos \theta + \sqrt{2} \sin \theta = \frac{\sqrt{3}}{2} \quad 0^\circ < \theta < 360^\circ$$

$$\sqrt{3} \sin(\theta + 35.26^\circ) = \frac{\sqrt{3}}{2}$$

$$\sin(\theta + 35.26^\circ) = \frac{1}{2}$$

$$0^\circ + 35.26^\circ < \theta + 35.26^\circ < 360^\circ + 35.26^\circ$$

$$35.26^\circ < \theta + 35.26^\circ < 395.26^\circ$$



$$\theta + 35.26^\circ = 180^\circ - 30^\circ; 360^\circ + 30^\circ$$

$$\theta + 35.26^\circ = 150^\circ; 390^\circ$$

$$\theta = 150^\circ - 35.26^\circ = 114.74^\circ; 390^\circ - 35.26^\circ$$

$$\theta = 114.74^\circ, 354.74^\circ$$

(b) Show the greatest value of $\frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5}$ is $\frac{5+\sqrt{3}}{22}$

$$\cos \theta + \sqrt{2} \sin \theta = \sqrt{3} \sin(\theta + 35.26^\circ)$$

$$-1 \leq \sin(\theta + 35.26^\circ) \leq 1$$

$$-\sqrt{3} \leq \sqrt{3} \sin(\theta + 35.26^\circ) \leq \sqrt{3}$$

$$-\sqrt{3} \leq \cos \theta + \sqrt{2} \sin \theta \leq \sqrt{3}$$

$$-\sqrt{3} + 5 \leq \cos \theta + \sqrt{2} \sin \theta + 5 \leq \sqrt{3} + 5$$

$$\frac{1}{\sqrt{3} + 5} \leq \frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5} \leq \frac{1}{-\sqrt{3} + 5}$$

$$\frac{1}{\sqrt{3} + 5} \leq \frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5} \leq \frac{1}{5 - \sqrt{3}}$$

The greatest value of $\frac{1}{\cos \theta + \sqrt{2} \sin \theta + 5}$:

$$\frac{1}{5 - \sqrt{3}} = \frac{1(5 + \sqrt{3})}{(5 - \sqrt{3})(5 + \sqrt{3})}$$

$$= \frac{5 + \sqrt{3}}{25 - 3}$$

$$= \frac{5 + \sqrt{3}}{22}$$

7. State the conditions for continuity of $f(x)$ at $x = a$.
- a. By using the conditions for continuity of $f(x)$ at $x = a$, find the values of m and n such that

$$f(x) = \begin{cases} n - 2 \cos x, & x < 0 \\ 2 + mx^2, & 0 \leq x < 2 \\ m - x, & x \geq 2 \end{cases}$$

is continuous on the interval $(-\infty, \infty)$.

- b. If $m = -2$ and $n = 4$, determine whether $f(x)$ is differentiable at $x = 2$ or not.

SOLUTION

The conditions for continuity of $f(x)$ at $x = a$

- i. $f(a)$ is defined
- ii. $\lim_{x \rightarrow a} f(x)$ is exist
- iii. $\lim_{x \rightarrow a} f(x) = f(a)$

$$(a) f(x) = \begin{cases} n - 2 \cos x, & x < 0 \\ 2 + mx^2, & 0 \leq x < 2 \\ m - x, & x \geq 2 \end{cases}$$

$f(x)$ is continuous at $x = 0$ and $x = 2$ as well.

When $x = 0$

$$\lim_{x \rightarrow 0^-} (n - 2 \cos x) = \lim_{x \rightarrow 0^+} (2 + mx^2)$$

$$n - 2 \cos(0) = 2 + m(0)^2$$

$$n - 2 = 2$$

$$n = 4$$

When $x = 2$

$$\lim_{x \rightarrow 2^-} (2 + mx^2) = \lim_{x \rightarrow 2^+} (m - x)$$

$$2 + m(2^2) = m - 2$$

$$3m + 4 = 0$$

$$m = -\frac{4}{3}$$

$$\therefore m = -\frac{4}{3}, n = 4$$

(b) If $m = -2$ and $n = 4$

$$f(x) = \begin{cases} 4 - 2 \cos x, & x < 0 \\ 2 - 2x^2, & 0 \leq x < 2 \\ -2 - x, & x \geq 2 \end{cases}$$

At $x = 2$

$$\begin{aligned} f'(2^-) &= \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{(2 - 2x^2) - (-2 - x)}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{2 - 2x^2 + 2 + x}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{-2x^2 + x + 4}{x - 2} \\ &= \frac{-2(2)^2 + (2) + 4}{2 - 2} \\ &= +\infty \text{ ((Undefined))} \end{aligned}$$

Since $f'(2^-)$ is undefined, therefore $f(x)$ is differentiable at $x = 2$.

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a^-) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a}$$

$$f'(a^+) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$$

8. A curve with equation $x^2 - 3y^2 = ae^{y-2x} + by - 6$, where a and b are constants, passes through the point $(1, 2)$.

a. Given $\frac{dy}{dx} = 1$ at $(1, 2)$, determine the values a and b .

b. Evaluate $\frac{d^2y}{dx^2}$ at $(1, 2)$.

SOLUTION

$$x^2 - 3y^2 = ae^{y-2x} + by - 6$$

At the point $(1, 2)$

$$1^2 - 3(2)^2 = ae^{2-2(1)} + b(2) - 6$$

$$1 - 12 = a + 2b - 6$$

$$a + 2b = -5 \quad \dots\dots\dots (1)$$

At the point $(1, 2)$, $\frac{dy}{dx} = 1$

$$x^2 - 3y^2 = ae^{y-2x} + by - 6$$

$$2x - 6y \frac{dy}{dx} = ae^{y-2x} \frac{d}{dx}(y - 2x) + b \frac{dy}{dx}$$

$$2x - 6y \frac{dy}{dx} = (ae^{y-2x}) \left[\frac{dy}{dx} - 2 \right] + b \frac{dy}{dx}$$

$$2(1) - 6(2)(1) = (ae^{(2)-2(1)})[1 - 2] + b(1)$$

$$-10 = -a + b$$

$$a - b = 10 \quad \dots\dots\dots (2)$$

$$(1) - (2)$$

$$3b = -15$$

$$b = -5$$

$$a + 5 = 10$$

$$a = 5$$

$$\therefore a = 5, \quad b = -5$$

(b) Evaluate $\frac{d^2y}{dx^2}$ at (1, 2)

$$x^2 - 3y^2 = 5e^{y-2x} - 5y - 6$$

$$2x - 6y \frac{dy}{dx} = (5e^{y-2x}) \left[\frac{dy}{dx} - 2 \right] - 5 \frac{dy}{dx}$$

$$2x - 6y \frac{dy}{dx} = 5e^{y-2x} \frac{dy}{dx} - 10e^{y-2x} - 5 \frac{dy}{dx}$$

$$2 - \left[6y \frac{d^2y}{dx^2} + 6 \left(\frac{dy}{dx} \right)^2 \right] = \left[5e^{y-2x} \frac{d^2y}{dx^2} + 5e^{y-2x} \frac{dy}{dx} \frac{d}{dx} (y - 2x) \right] - 10e^{y-2x} \frac{d}{dx} (y - 2x) - 5 \frac{d^2y}{dx^2}$$

$$2 - \left[6y \frac{d^2y}{dx^2} + 6 \left(\frac{dy}{dx} \right)^2 \right] = \left[5e^{y-2x} \frac{d^2y}{dx^2} + 5e^{y-2x} \frac{dy}{dx} \left(\frac{dy}{dx} - 2 \right) \right] - 10e^{y-2x} \left(\frac{dy}{dx} - 2 \right) - 5 \frac{d^2y}{dx^2}$$

At the point (1, 2), $\frac{dy}{dx} = 1$

$$2 - \left[6y \frac{d^2y}{dx^2} + 6 \left(\frac{dy}{dx} \right)^2 \right] = \left[5e^{y-2x} \frac{d^2y}{dx^2} + 5e^{y-2x} \frac{dy}{dx} \left(\frac{dy}{dx} - 2 \right) \right] - 10e^{y-2x} \left(\frac{dy}{dx} - 2 \right) - 5 \frac{d^2y}{dx^2}$$

$$2 - \left[6(2) \frac{d^2y}{dx^2} + 6(1)^2 \right] = \left[5e^{2-2(1)} \frac{d^2y}{dx^2} + 5e^{2-2(1)}(1)((1) - 2) \right] - 10e^{2-2(1)}((1) - 2) - 5 \frac{d^2y}{dx^2}$$

$$2 - \left[12 \frac{d^2y}{dx^2} + 6 \right] = \left[5 \frac{d^2y}{dx^2} - 5 \right] + 10 - 5 \frac{d^2y}{dx^2}$$

$$2 - 12 \frac{d^2y}{dx^2} - 6 = 5 \frac{d^2y}{dx^2} - 5 + 10 - 5 \frac{d^2y}{dx^2}$$

$$5 \frac{d^2y}{dx^2} + 12 \frac{d^2y}{dx^2} - 5 \frac{d^2y}{dx^2} = 2 + 5 - 10 - 6$$

$$12 \frac{d^2y}{dx^2} = -9$$

$$\frac{d^2y}{dx^2} = -\frac{9}{12}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{4}$$

9. The function f is defined by $f(x) = \frac{\ln(x-1)}{x-1}$ for $x > 1$.
- By considering the first and second derivatives of $f(x)$, show that there is only one maximum point on the graph $y = f(x)$.
 - Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.
 - Find the x -coordinate of the function f when $\frac{d^2y}{dx^2} = 0$.

SOLUTION

$$(a) f(x) = \frac{\ln(x-1)}{x-1}, \quad \text{for } x > 1$$

$$u = \ln(x-1) \qquad v = x-1$$

$$u' = \frac{1}{x-1} \frac{d}{dx}(x-1) \qquad v' = 1$$

$$= \frac{1}{x-1}$$

$$f'(x) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x-1)\left(\frac{1}{x-1}\right) - [\ln(x-1)](1)}{(x-1)^2}$$

$$= \frac{1 - \ln(x-1)}{(x-1)^2}$$

$$\text{Let } f'(x) = 0$$

$$\frac{1 - \ln(x-1)}{(x-1)^2} = 0$$

$$1 - \ln(x-1) = 0$$

$$\ln(x-1) = 1$$

$$x-1 = e^1$$

$$x = e + 1$$

$$\log_a b = c \Leftrightarrow b = a^c$$

$$\ln b = c \Leftrightarrow b = e^c$$

$$f'(x) = \frac{1 - \ln(x-1)}{(x-1)^2}$$

$$u = 1 - \ln(x-1)$$

$$v = (x-1)^2$$

$$u' = -\frac{1}{x-1}$$

$$v' = 2(x-1)$$

$$\begin{aligned} f''(x) &= \frac{(x-1)^2 \left(-\frac{1}{x-1}\right) - [1 - \ln(x-1)]2(x-1)}{[(x-1)^2]^2} \\ &= \frac{[-(x-1)] - [1 - \ln(x-1)](2x-2)}{(x-1)^4} \end{aligned}$$

When $x = e + 1$

$$\begin{aligned} f''(x) &= \frac{(-(e+1-1)) - [1 - \ln(e+1-1)][(2(e+1)-2)]}{(e+1-1)^4} \\ &= \frac{(-e) - [1 - \ln(e)](2e)}{(e)^4} \\ &= \frac{(-e)}{(e)^4} \\ &= -\frac{1}{e^3} < 0 \text{ (Maximum)} \end{aligned}$$

- (b) Use the result obtained in part 9(a) to state the exact coordinates of the maximum point.

$$f(x) = \frac{\ln(x-1)}{x-1}$$

When $x = e + 1$

$$\begin{aligned} f(x) &= \frac{\ln(e+1-1)}{e+1-1} \\ &= \frac{\ln e}{e} \\ &= \frac{1}{e} \end{aligned}$$

\therefore the exact coordinates of the maximum point: $\left(e + 1, \frac{1}{e}\right)$

(c) Find the x -coordinate of the function f when $\frac{d^2y}{dx^2} = 0$

$$\text{When } \frac{d^2y}{dx^2} = 0$$

$$\frac{[-(x-1)] - [1 - \ln(x-1)](2x-2)}{(x-1)^4} = 0$$

$$[-(x-1)] - [1 - \ln(x-1)](2x-2) = 0$$

$$[-(x-1)] - 2(x-1)[1 - \ln(x-1)] = 0$$

$$(x-1)[-1 - 2(1 - \ln(x-1))] = 0$$

$$(x-1)[-1 - 2 + 2\ln(x-1)] = 0$$

$$(x-1)[-3 + 2\ln(x-1)] = 0$$

$$(x-1) = 0 \qquad -3 + 2\ln(x-1) = 0$$

$$x = 1 \qquad 2\ln(x-1) = 3$$

$$\ln(x-1) = \frac{3}{2}$$

$$x-1 = e^{\frac{3}{2}}$$

$$x = e^{\frac{3}{2}} + 1$$

Since $x \neq 1$, therefore $x = e^{\frac{3}{2}} + 1$

10. A curve is defined by the parametric equations $x = 3t - \frac{1}{t}$ and $y = t + \frac{3}{t}$, where $t \neq 0$.

a. Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$.

b. Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that

$$-3 < \frac{dy}{dx} < \frac{1}{3}.$$

SOLUTION

$$x = 3t - \frac{1}{t} \quad \text{and} \quad y = t + \frac{3}{t}$$

(a) Show that $\frac{dy}{dx} = \frac{t^2-3}{3t^2+1}$. Hence, find $\frac{d^2y}{dx^2}$

$$x = 3t - \frac{1}{t}$$

$$y = t + \frac{3}{t}$$

$$\frac{dx}{dt} = 3 + \frac{1}{t^2}$$

$$\frac{dy}{dt} = 1 - \frac{3}{t^2}$$

$$= \frac{3t^2 + 1}{t^2}$$

$$= \frac{t^2 - 3}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{t^2 - 3}{t^2} \cdot \frac{t^2}{3t^2 + 1}$$

$$= \frac{t^2 - 3}{3t^2 + 1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{t^2 - 3}{3t^2 + 1}$$

$$u = t^2 - 3$$

$$v = 3t^2 + 1$$

$$u' = 2t$$

$$v' = 6t$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{vu' - uv'}{v^2} \\ &= \frac{(3t^2 + 1)(2t) - (t^2 - 3)(6t)}{(3t^2 + 1)^2} \\ &= \frac{6t^3 + 2t - 6t^3 + 18t}{(3t^2 + 1)^2} \\ &= \frac{20t}{(3t^2 + 1)^2} \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= \frac{20t}{(3t^2 + 1)^2} \cdot \frac{t^2}{3t^2 + 1} \\ &= \frac{20t^3}{(3t^2 + 1)^3} \end{aligned}$$

(b) Show that $\frac{dy}{dx}$ can be expressed as $\frac{dy}{dx} = \frac{1}{3} - \frac{10}{3(3t^2+1)}$. Hence, deduce that

$$-3 < \frac{dy}{dx} < \frac{1}{3}.$$

$$\frac{dy}{dx} = \frac{t^2 - 3}{3t^2 + 1}$$

$$\begin{array}{r} \frac{1}{3} \\ 3t^2 + 1 \overline{) t^2 - 3} \\ \underline{t^2 + \frac{1}{3}} \\ -\frac{10}{3} \end{array}$$

$$\frac{dy}{dx} = \frac{1}{3} + \frac{-\frac{10}{3}}{3t^2 + 1}$$

$$\frac{1}{3} - \frac{10}{3(3t^2 + 1)}$$

$$t^2 > 0$$

$$3t^2 > 0$$

$$3t^2 + 1 > 0 + 1$$

$$3t^2 + 1 > 1$$

$$3(3t^2 + 1) > 3$$

$$3 < 3(3t^2 + 1) < \infty$$

$$0 < \frac{1}{3(3t^2 + 1)} < \frac{1}{3}$$

$$0 < \frac{10}{3(3t^2 + 1)} < \frac{10}{3}$$

$$-\frac{10}{3} < -\frac{10}{3(3t^2 + 1)} < 0$$

$$\frac{1}{3} - \frac{10}{3} < \frac{1}{3} - \frac{10}{3(3t^2 + 1)} < \frac{1}{3} + 0$$

$$-3 < \frac{1}{3} - \frac{10}{3(3t^2 + 1)} < \frac{1}{3}$$

$$-3 < \frac{dy}{dx} < \frac{1}{3}$$