KEMENTERIAN PENDIDIKAN MALAYSIA

SM025/1

Matriculation Programme Examination

Semester 2

Session 2018/2019

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1. The size of a population of insects is increasing at a rate proportional to the number of insects, N, in time t days which satisfies the equation $\frac{dN}{dt} = kN$, where $k > 0$. Given that the number of insects at the beginning of an observation is N_0 and is double in 2 days, find the number of insects after 5 days.
2. Sketch and shade the region bounded by the curve $y = 4e^{-x}$, the straight line $y = 4 - x, y - axis\ and\ x = 3$. Hence, find the area of the shaded region by using trapezoidal rule with five ordinates. Give your answer correct to four decimal places.
3. Given a circle $x^2 + y^2 + kx + 6y + 8 = 0$, where k is a positive constant.
   a) Determine the value of k and the centre of the circle if the radius is $\frac{\sqrt{13}}{2}$ unit.
   b) Find the points of intersection of the circle with straight line $y - x + 2 = 0$. Hence, obtain one of the tangent equation at the point of intersection.
4. The continuous random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right) & 0 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

   a) Find the median
   b) Determine the probability density function of X.
   c) Hence, find the mode and the mean.
   d) State the skewness of the distribution with a reason.
5. The amount of cement packed by a machine is normally distributed with mean 39.3kg and standard deviation 0.9kg. A bag of cement is randomly selected.
   a) Find the probability that the bag weighs more than 40kg.
   b) If the probability of the bag weighs not more than m kg is 0.95, determine the value of m.
   c) A total of 5 bags of cement are chosen at random. Find the probability that at least 4 bags weigh more than 40kg.

END OF QUESTION PAPER

1. The size of a population of insects is increasing at a rate proportional to the number of insects, N, in time t days which satisfies the equation $\frac{dN}{dt} = kN$, where $k > 0$. Given that the number of insects at the beginning of an observation is N_0 and is double in 2 days, find the number of insects after 5 days.

SOLUTION

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = kdt$$

$$\int \frac{dN}{N} = \int kdt$$

$$\ln N = kt + C$$

$$N = e^{kt+c}$$

$$N = Ae^{kt}$$

Given that when

$t = 0; \quad N = N_o$

$$N_o = Ae^{k(0)}$$

$$A = N_o$$

$t = 2; \quad N = 2N_o, \qquad A = N_o$

$$2N_o = N_o e^{k(2)}$$

$$e^{2k} = \frac{2N_o}{N_o}$$

$$e^{2k} = 2$$

$$2k = \ln 2$$

$$k = \frac{\ln 2}{2} = 0.3466$$

$$N = N_o e^{0.3466t}$$

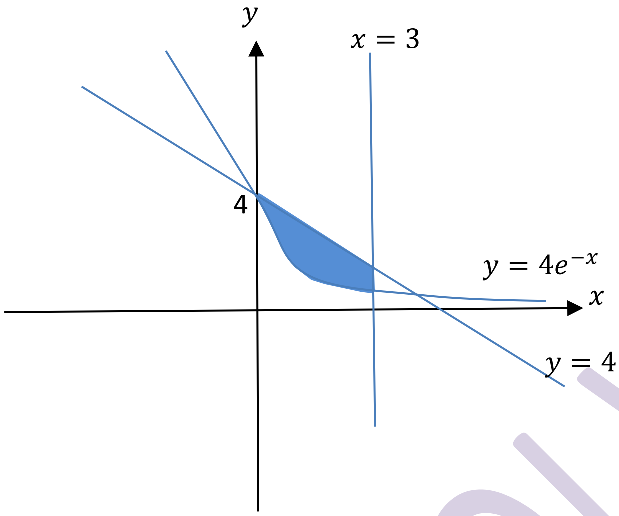
$When\ t = 5$:

$$N = N_o e^{0.3466(5)}$$

$$= 5.66N_o$$

2. Sketch and shade the region bounded by the curve $y = 4e^{-x}$, the straight line $y = 4 - x$, $y - axis\ and\ x = 3$. Hence, find the area of the shaded region by **using trapezoidal rule** with five ordinates. Give your answer correct to four decimal places.

SOLUTION



$$Area = \int_0^3 (4 - x) - (4e^{-x})\, dx$$

$Given\ n = 4$

$$h = \frac{3 - 0}{4} = 0.75$$

	$y = 4 - x - 4e^{-x}$	
$x_0 = 0$	0	
$x_1 = 0.75$		1.36053
$x_2 = 1.5$		1.60748
$x_3 = 2.25$		1.32840
$x_4 = 3.0$	0.80085	
Total	0.80085	4.29641

$$Area = \frac{h}{2}[(x_0 + x_4) + 2(x_1 + x_2 + x_3)]$$

$$= \frac{0.75}{2}[0.80085 + 2(4.29641)]$$

$$= 3.5226\ unit^2$$

3. Given a circle $x^2 + y^2 + kx + 6y + 8 = 0$, where k is a positive constant.

a) Determine the value of k and the centre of the circle if the radius is $\frac{\sqrt{13}}{2}$ unit.

b) Find the points of intersection of the circle with straight line $y - x + 2 = 0$. Hence, obtain one of the tangent equations at the point of intersection.

SOLUTION

Equation of Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Where

$$r = \sqrt{f^2 + g^2 - c}$$

$$Center, C = (-g, -f)$$

(3a)

$$x^2 + y^2 + kx + 6y + 8 = 0$$

$2g = k \qquad 2f = 6 \qquad c = 8$

$g = \frac{k}{2} \qquad f = 3$

$$r = \sqrt{f^2 + g^2 - c}$$

$$\frac{\sqrt{13}}{2} = \sqrt{3^2 + \left(\frac{k}{2}\right)^2 - 8}$$

$$\frac{\sqrt{13}}{2} = \sqrt{1 + \frac{k^2}{4}}$$

$$\frac{13}{4} = 1 + \frac{k^2}{4}$$

$$\frac{k^2}{4} = \frac{13}{4} - 1$$

$$\frac{k^2}{4} = \frac{9}{4}$$

$$k^2 = 9$$

$$k = 3 \ (k > 0)$$

$$\textbf{Center of the circle} = (-g, -f) = \left(-\frac{3}{2}, -3\right)$$

(3b)

Equation of circle

$x^2 + y^2 + 3x + 6y + 8 = 0$ .... (1)

Equation of straight line

$y - x + 2 = 0$

$y = x - 2$ .... (2)

Substitute (2) into (1)

$$x^2 + (x-2)^2 + 3x + 6(x-2) + 8 = 0$$

$$x^2 + x^2 - 4x + 4 + 3x + 6x - 12 + 8 = 0$$

$$2x^2 + 5x = 0$$

$$x(2x+5) = 0$$

$$x = 0 \quad or \quad x = -\frac{5}{2}$$

$$y = -2 \quad or \quad y = -\frac{9}{2}$$

Therefore the intersection points are $(0, -2)$ *and* $\left(-\frac{5}{2}, -\frac{9}{2}\right)$.

Equation of tanget at $(\mathbf{0}, \mathbf{-2})$ ***for*** $\mathbf{x^2 + y^2 + 3x + 6y + 8 = 0}$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$x_1 = 0;\ y_1 = -2;\ g = \frac{3}{2};\ f = 3, c = 8$

$$x(0) + y(-2) + \frac{3}{2}(x+0) + 3(y-2) + 8 = 0$$

$$-2y + \frac{3}{2}x + 3y - 6 + 8 = 0$$

$$y + \frac{3}{2}x + 2 = 0$$

$$2y + 3x + 4 = 0$$

or

$\boldsymbol{Equation\ of\ tanget\ at\left(-\frac{5}{2},-\frac{9}{2}\right)for\ x^2+y^2+3x+6y+8=0}$

$x_1=-\frac{5}{2};\ \ y_1=-\frac{9}{2};\ g=\frac{3}{2};\ f=3, c=8$

$$x\left(-\frac{5}{2}\right)+y\left(-\frac{9}{2}\right)+\frac{3}{2}\left(x-\frac{5}{2}\right)+3\left(y-\frac{9}{2}\right)+8=0$$

$$-\frac{5}{2}x-\frac{9}{2}y+\frac{3}{2}x-\frac{15}{4}+3y-\frac{27}{2}+8=0$$

$$-10x-18y+6x-15+12y-54+32=0$$

$$-4x-6y-37=0$$

$$4x+6y+37=0$$

4. The continuous random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right) & 0 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

a) Find the median

b) Determine the probability density function of X.

c) Hence, find the mode and the mean.

d) State the skewness of the distribution with a reason.

SOLUTION

(4a)

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right) & 0 \le x \le 3 \\ 1, & x \ge 3 \end{cases}$$

$Median$:

$$F(m) = \frac{1}{2}$$

$$\frac{1}{9}\left(2m^2 - \frac{m^3}{3}\right) = \frac{1}{2}$$

$$2m^2 - \frac{m^3}{3} = \frac{9}{2}$$

$$12m^2 - 2m^3 = 27$$

$$2m^3 - 12m^2 + 27 = 0$$

From Calculator

$m = 5.564 \quad or \quad m = 1.7907 \quad or \quad m = -1.3548$

$\therefore median = 1.7907$

(4b)

$x \leq 0$ $\qquad f(x) = \frac{d}{dx}(0) = 0$

$0 \leq x \leq 3$ $\qquad f(x) = \frac{d}{dx}\frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right)$

$$= \frac{1}{9}(4x - x^2)$$

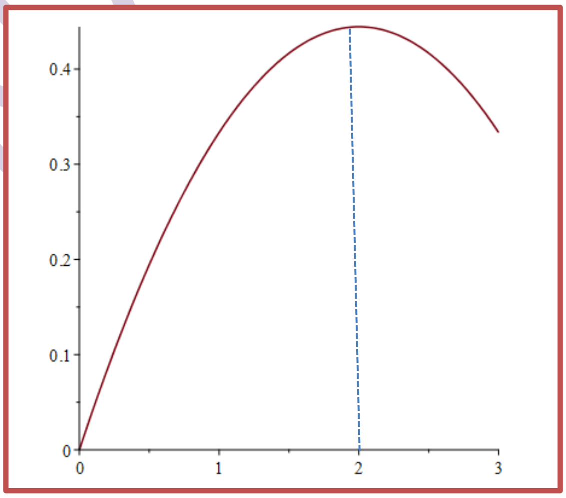
$$= \frac{4}{9}x - \frac{1}{9}x^2$$

$x \geq 3$ $\qquad f(x) = \frac{d}{dx}(1) = 0$

$$f(x) = \begin{cases} \frac{4}{9}x - \frac{1}{9}x^2 & , \quad 0 \leq x \leq 3 \\ 0 & , \quad otherwise \end{cases}$$

(4c)

$$f(x) = \frac{4}{9}x - \frac{1}{9}x^2$$



$$a = -\frac{1}{9}, b = \frac{4}{9}; c = 0$$

$Maximum\ point$:

$$x = -\frac{b}{2a}$$

$$= -\frac{\frac{4}{9}}{2\left(-\frac{1}{9}\right)}$$

$$= 2$$

$\therefore Mode: x = 2$

$$Mean = E(x) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$Mean = E(x) = \int_{-\infty}^{0} x\,(0)dx + \int_{0}^{3} x\left(\frac{4}{9}x - \frac{1}{9}x^2\right)dx + \int_{3}^{\infty} x\,(0)dx$$

$$= \int_0^3 x\left(\frac{4}{9}x - \frac{1}{9}x^2\right) dx$$

$$= \int_0^3 \frac{4}{9}x^2 - \frac{1}{9}x^3 dx$$

$$= \left[\frac{4}{27}x^3 - \frac{1}{36}x^4\right]_0^3$$

$$= \left(\frac{4}{27}3^3 - \frac{1}{36}3^4\right) - (0)$$

$$= 4 - \frac{9}{4}$$

$$= \frac{7}{4}$$

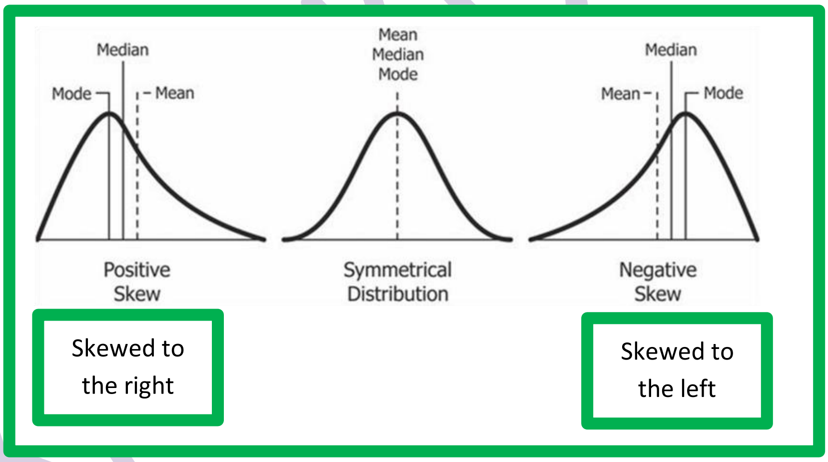
(4d)

$Since\ Mean < Mode, therefore\ the\ skewness\ is\ skewed\ to\ the\ left.$

or

$Since\ Mean < median, therefore\ the\ skewness\ is\ skewed\ to\ the\ left.$

Note:



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SOLUTION

(5a)

$\mu = 39.3 \qquad \sigma = 0.9$

$X \sim N(39.3, 0.9^2)$

$$P(X > 40) = P\left(Z > \frac{40 - 39.3}{0.9}\right)$$
$$= P(Z > 0.78)$$
$$= 0.2177$$

(5b)

$$P(X < m) = 0.95$$
$$P\left(Z < \frac{m - 39.3}{0.9}\right) = 0.95$$
$$P\left(Z \geq \frac{m - 39.3}{0.9}\right) = 0.05$$

From statiscal table:

$$(Z \geq 1.65) = 0.05$$
$$\frac{m - 39.3}{0.9} = 1.65$$
$$m = 40.785$$

(5c)

$$X \sim B(5, 0.2177)$$
$$P(X \geq 4) = P(X = 4) + P(x = 5)$$
$$= {}^5C_4(0.2177)^4(0.7826)^1 + {}^5C_5(0.2177)^5(0.7826)^0$$
$$= 0.00928$$